LETTER TO THE EDITOR

Oscillations in the frequency dependence of long-range correlations of waves

N Shnerb and M Kaveh
Department of Physics, Bar-Ilan University, Ramat-Gan 52100, Israel

Received 17 August 1990

Abstract. We predict new oscillations in the frequency dependence of the intensity autocorrelation functions of waves for a tube geometry. We have performed numerical simulations which clearly confirm the existence of new long-range correlations.

Recently, much work has been devoted [1-7] to the study of intensity correlations of waves in random systems. Quite remarkably, these correlations are of a long-range nature and enhance [1] the fluctuations of the transmission coefficient in a non-classical manner [1,3]. That is, \(\langle T^2 \rangle - \langle T \rangle^2\) is not proportional to \(\Omega^{-1}\) (where \(\Omega\) is the volume) but depends explicitly on the length \(L\) (in a slab geometry) and the width \(W\) of the system.

In this letter, we study the frequency dependence of the intensity autocorrelation function \(C(\Delta \omega)\) for a tube geometry, where \(L \gg W\). Recently, this geometry has become interesting experimentally [6,7]. We find new behaviour for \(C(\Delta \omega)\) for \(\Delta \omega > D/W^2\) (where \(D\) is the diffusion constant). In this region, \(C(\Delta \omega)\) oscillates as \(\Delta \omega\) increases with an amplitude which decays slowly as \(\Delta \omega^{-1}\). We have also performed numerical simulations by applying the Edrei-Kaveh [8] method to this new geometry and have found new long-range behaviour.

The long-range nature of the intensity fluctuations was first demonstrated for their angular dependence. This was achieved by two independent methods: by diagrammatic techniques [1,2] and by the Langevin approach [4]. Both methods seem to coincide (up to numerical factors [4]). The angular correlation function is defined as

\[
C(\Delta \hat{q}_a, \Delta \hat{q}_b) \equiv \langle \delta I(\hat{q}_a, \hat{q}_b) \delta I(\hat{q}_a', \hat{q}_b') \rangle
\]

where \(\delta I(\hat{q}_a, \hat{q}_b) = I(\hat{q}_a, \hat{q}_b) - \langle I(\hat{q}_a, \hat{q}_b) \rangle\) and similarly for \(\delta I(\hat{q}_a', \hat{q}_b')\). The vectors \(\hat{q}_a, \hat{q}_b\) correspond to the incident and emitted wavevector, respectively. For two given such pairs of wavevectors \((\hat{q}_a, \hat{q}_b)\) and \((\hat{q}_a', \hat{q}_b')\), it was shown that \(C\) is a function of \(\Delta \hat{q}_a = \hat{q}_a - \hat{q}_a'\) and \(\Delta \hat{q}_b = \hat{q}_b - \hat{q}_b'\). Equation (1) can be expanded [2] in powers of the inverse dimensionless conductance \(g^{-1} = (\lambda/W)^{d-1}(L/l)\), where \(l\) is the elastic transport mean free path and \(d\) is the dimensionality of the system. Thus, (1) can be written as

\[
C(\Delta \hat{q}_a, \Delta \hat{q}_b) = \sum_n g^{1-n} C_n(\Delta \hat{q}_a, \Delta \hat{q}_b).
\]

Until now, only the first three terms have been calculated. For wide samples for which \(W \gg L\), \(g^{-1}\) is extremely small which causes \(C_2\) and \(C_3\) to be almost unobservable. The short-range contribution \(C_1\) is called the ‘memory effect’ [2,9,10]. \(C_1\) was readily...
determined numerically [5] and observed experimentally [10]. The main point to note is that \( C_2 \) is identical for a two-dimensional system and a three-dimensional system. It is therefore easier to detect \( C_2 \) for a two-dimensional system because \( g^{-1} \) is then larger by a factor \( W/\lambda \). Indeed, \( C_2(\Delta \tilde{q}_a, \Delta \tilde{q}_b) \) has recently been obtained from numerical simulations [11] and found to be in agreement with the analytical predictions. \( C_3 \) has not yet been determined for a wide slab for which \( W \gg L \).

By analogy with (1), one may define the frequency-dependent autocorrelation function

\[
C(\Delta \omega) = \langle \delta I(\omega) \delta I(\omega + \Delta \omega) \rangle
\]

where \( \delta I(\omega) = I(\omega) - \langle I(\omega) \rangle \). This can also be expanded in powers of \( g^{-1} \),

\[
C(\Delta \omega) = \sum_n g^{1-n} F_n(\Delta \omega).
\]

For wide samples for which \( W \gg L \), only \( F_1(\Delta \omega) \) has been determined numerically [11] or experimentally [12]. The long-range correlations, \( F_2(\Delta \omega) \) and \( F_3(\Delta \omega) \) contribute negligibly to \( C(\Delta \omega) \). Analytical expressions for \( F_1(\Delta \omega) \) were obtained [13] for different geometries by showing [13] that

\[
F_1(\Delta \omega) = |\int P(t) \exp(i\Delta \omega t) \, dt|^2
\]

where \( P(t) \) is the diffusive probability for a multiple-scattering trajectory of length \( Vt \) (where \( V \) is the velocity of light in the medium). Equation (5) has been obtained by Edrei and Kaveh [13] and confirmed by numerical simulations [8]. Genack and Drake determined [12] \( P(t) \) experimentally from the transmitted pulse shape of a slab and showed that the resultant \( F_1(\Delta \omega) \) (from (5)) is in excellent agreement with the measured \( C(\Delta \omega) \). This confirms that the contributions of the higher-order terms in (4), \( F_2(\Delta \omega) \) and \( F_3(\Delta \omega) \), are negligible when \( W \gg L \).

Recently, van Albada and Lagendijk [7] showed that when a point source is used instead of a plane wave, the contribution of \( F_2(\Delta \omega) \) to the total transmission coefficient is enhanced. They were able to determine \( F_2(\Delta \omega) \) and showed that it agrees with the calculation of Pini and Shapiro [4].

Genack and co-workers [6] has recently pointed out that by using a tube geometry, one enhances the contributions of \( F_2 \) and \( F_3 \) because \( g^{-1} \) becomes rather larger when \( W \ll L \).

The purpose of this letter is to show that the functional form for a tube geometry is entirely different from the wide geometry, for which \( W \gg L \). We show that \( F_2(\Delta \omega) \) depends markedly on \( W/L \). For \( W/L \ll 1 \), we find that \( F_2(\Delta \omega) \) is almost independent of \( \Delta \omega \) and contributes a constant correlation, similar to \( F_3 \). Thus, in this regime \( F_2 \) is indistinguishable from the constant \( F_3 \) except that its contribution to \( C(\Delta \omega) \) is larger by a factor \( g \).

When \( W \ll L \), we find for \( F_2(\Delta \omega) \),

\[
F_2(\Delta \omega) = \left( \frac{\sin x}{x} \right)^2
\]

where \( x = (\Delta \omega / D)^{1/2} W/2 \). We see that, unlike the case of a wide system \( W \gg L \), \( F_2(\Delta \omega) \) is independent of the length of the system \( L \). This is in sharp contrast to \( F_1(\Delta \omega) \) in this geometry. We can show that, to an excellent approximation, \( F_1(\Delta \omega) \) is independent of \( W \) and continues to scale as \( (\Delta \omega / D)^{1/2} L \). To prove this result, we turn to (5) and show that \( P(t) \) is almost independent on \( W \). For a given \( W \), we have to impose an additional
two reflecting boundary conditions (we use a two-dimensional tube). In this case, we get $P_w(t) = P(t)\Delta(t, y)$, where $P(t)$ corresponds to $W \to \infty$ and $\Delta(t, y)$ is the correction due to a finite $W$. In this case, $P_w(t)$ depends on the distance $y$ from the side boundary. Solving the diffusion equation with the above boundary conditions yields

$$\Delta(t, y) = \sum_{n=-\infty}^{\infty} n^{-1} \sin \left( \frac{2\pi ny_0}{W} \right) \cos \left( \frac{2\pi ny}{W} \right) \exp \left( -\pi^2 D n^2 t/W^2 \right) \quad (7)$$

where $y_0$ is a negligibly small number ($y_0 < W$) which is introduced to avoid conflicting boundary conditions at the corners $y = 0$ and $y = W$. As long as $W > l$, we must have $y_0 < W$ and only the $n = 0$ term in (7) contributes, leading to $\Delta(t, y) = 1$. This result is in striking contrast to $C_1(\Delta q_{ao}, \Delta q_{bs})$ for a tube geometry which was recently shown by Eliyahu et al [14] to differ significantly from a wide system where $W \gg L$. From this result we may conclude that even for a tube geometry (where $g^{-1}$ is not small), $F_2(\Delta o)$ may be observed only for $\Delta \omega > D/L^2$, where $F_1(\Delta o)$ is small. For $\Delta \omega$ near the half-width $\Delta \omega_{hw} = D/L^2$, we get for $x$ in (6), $x = W/2L$. Since for a tube geometry $W/L < 1$, we find $F_2(\Delta \omega \lesssim D/L^2) \approx 1$. Thus, $F_2(\Delta o)$ is almost constant and indistinguishable from $F_3$. When $\Delta \omega$ increases much above $\Delta \omega_{hw}$, $F_1(\Delta o)$ decays exponentially and $F_2(\Delta o)$ is revealed. When $\Delta \omega > D/W^2$, $F_2(\Delta o)$ oscillates according to equation (6). Thus, the range of frequencies where these oscillations should be observed is $\Delta \omega \gtrsim \Delta \omega_{hw}(L/W)^2$. We may distinguish two cases. For tubes where $L \approx W$, the oscillatory region is difficult to reach and $F_2(\Delta o) = 1$. For tubes where $L \gtrsim W$, we predict that near the tail of $C(\Delta o)$ (for $\Delta \omega > \Delta \omega_{hw}$), oscillatory behaviour should set in.

We have performed numerical simulations to verify these predictions, using the Edrei–Kaveh method [8] for calculation $C(\Delta o)$. We first show the results for a two-dimensional tube with $L = 7W$. In this case, oscillatory behaviour should be observed only for $\Delta \omega > 49\Delta \omega_{hw}$. In figure 1, we show our numerical results and compare them with the theory. All the results are normalized to $C(\Delta o = 0)$. The squares represent $F_1(\Delta o)$. It should be emphasized that $F_1(\Delta o)$ is calculated numerically directly from the electric field/electric field correlation, $F_1(\Delta o) \equiv |(E^*(\omega)E(\omega + \Delta o))|^2$. We compare our simulations with $F_1(\Delta o)$ as calculated from (5). The excellent agreement is evident from the figure. In figure 1, we also show the entire correlation function $C(\Delta o)$ which is represented by the plus signs. The fact that $C(\Delta o)$ is always larger than $F_1(\Delta o)$ is due to the long-range contributions $F_2(\Delta o)$ and $F_3(\Delta o)$. For this geometry, we plot $C_2(\Delta o) = g^{-1}F_2(\Delta o)$ as given by (5). For the range of frequencies in the figure, $F_2(\Delta o) = 1$ and $C_2(\Delta o) = g^{-1} = 0.28$. We also plot $C_3(\Delta o) = g^{-2}F_3(\Delta o)$. Since $F_3(\Delta o) = 1$, $C_3(\Delta o) = g^{-2} = 0.078$. The broken curve in figure 1 represents the total contribution $C(\Delta o) = C_1(\Delta o) + C_2(\Delta o) + C_3(\Delta o)$. We see that there is excellent agreement between the analytical results for $C(\Delta o)$ and those obtained by the simulations (the plus signs). Figure 1 confirms our prediction that for $L \gg W$, $F_2(\Delta o) = 1$ and is similar to $F_3(\Delta o) = 1$. Of course, the contribution of $F_2(\Delta o)$ to $C(\Delta o)$ is larger than $F_3(\Delta o)$ by a factor $g = 3.6$.

We now turn to shorter tubes in order to study the existence of the oscillations of $F_2(\Delta o)$. We have used a two-dimensional tube with $L = 2W$. Here we expect to see oscillations for $\Delta \omega \gtrsim 4\Delta \omega_{hw}$. In figure 2, we plot $C(\Delta o)$ as a function of $\Delta o$. The broken curve is $F_1(\Delta o)$ which agrees with the simulations for $\Delta \omega < \Delta \omega_{hw}$. For $\Delta \omega > \Delta \omega_{hw}$, $F_1(\Delta o)$ decays exponentially and $C(\Delta o) = C_2(\Delta o) + C_3$. The full curve corresponds to $C_2(\Delta o)$ where $C_2(\Delta o) = g^{-1}F_2(\Delta o) = 0.09|\sin x/|x||^2$ with $x = (\Delta \omega/D)^{1/2}(W/2)$ and $C_3 = g^{-2} = 0.0081$. We see that in this regime, $C_2(\Delta o)$ makes the dominant contribution to $C(\Delta o)$. The analytical result $F_2(\Delta o)$ was calculated
Figure 1. $C(\Delta \omega)$ as a function of $\Delta \omega/\omega$ for $L = 7W$. The squares represent the numerical results for $F_1(\Delta \omega)$ and the full curve the analytical results as extracted from (5). The plus signs represent the numerical results for $C(\Delta \omega)$ and the broken curve the analytic results (see text). Curve (a) represents $C_3(\Delta \omega)$ as given by (6) and curve (b) represents $C_3 = g^{-2}$.

Figure 2. $C(\Delta \omega)$ as a function of $\Delta \omega/\omega$ for $L = 2W$. The full curve is the calculated $C(\Delta \omega)$ and the broken curve is $F_1(\Delta \omega)$ as extracted from (5).
as follows. By using the Hikami box diagrams [15], Eliyahu et al [14] have recently calculated $C_2(\Delta q_a, \Delta q_b)$ for a tube geometry. Setting $\Delta q_a = 0$, we get $C_2(\Delta q_a = 0, \Delta q_b) = [\sin(\Delta q_b W/2)/(\Delta q_b W/2)]^2$. When a frequency shift $\Delta \omega$ is introduced in the propagators, it serves as a cut-off and $\Delta q_b$ must be replaced by $(\Delta \omega / D)^{1/2}$. This leads to (6). The numerical simulations were not accurate enough to follow this oscillatory behaviour.

In summary, we have shown that the long-range contribution $F_2(\Delta \omega)$ for a tube geometry is entirely different from that of a wide system where $W \gg L$. For tubes where $L \gg W$, $F_2(\Delta \omega)$ is a constant, independent of $\Delta \omega$ for a wide range, $\Delta \omega < D / W^2$. For $\Delta \omega > D / W^2$, $F_2(\Delta \omega)$ has an oscillatory character and dominates $C(\Delta \omega)$.

We acknowledge important discussions with Richard Berkovits. This work was supported by the United States–Israel Binational Science Foundation (BSF).

References

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