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Gauge and group properties of massless fields in any dimension

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Abstract. It is shown that the solvable structure of the stability group of a massless field yields, in a simple and short way, useful information about the physical polarization states, and the structure of the Hilbert space for such a field. It is also shown that such fields are necessarily gauge fields; the exact form of the gauge transformation follows from the structure of the equivalence classes implied by the group properties. Some examples for spinor and tensor fields in general dimensions are worked out and the structure of the Hilbert space for gravitational radiation is discussed in some detail.

1. Introduction

The relations between the physical properties of the four-dimensional massless fields and the representation of the Poincaré group have been treated by many authors [1–3]. It appears that in most cases in which this subject is treated, it is *assumed* that the field has a definite structure, e.g. that the massless vector field is a *gauge field.* Weinberg [4] has, however, treated this problem without assuming that the massless fields are gauge fields, and without specific assumptions on the form of the equations of motion. He starts by defining tensor fields transforming according to the $(0, j)$ or $(j, 0)$ representations of the homogeneous Lorentz group (corresponding, for $j = 1$ and 2, to the Maxwell field strengths $F^{\mu\nu}$ and the Riemann–Christofel tensor $R^{\mu\nu\lambda\sigma}$). He shows that any covariant-free field can be constructed as a linear combination of these fields and their derivatives. They cannot, however, be used to construct the interaction Hamiltonian, since the coefficients of the annihilation and creation operators for particles of momentum p and spin j vanish as p^j for $p \to 0$, inconsistent with the existence of long-range force laws. One must therefore use potentials. It is then shown explicitly that the vector potentials do not transform as tensors under Lorentz transformation. A Lorentz transformation induces a tensor transformation and additional terms which are derivatives of a scalar function of x and Λ; these additional terms are understood as a gauge transformation, since they leave the field strength invariant. The only interactions allowed are, therefore, those which satisfy gauge invariance, i.e. they must be coupled to conserved currents. One finds, moreover, that a non-covariant term must be added to the interaction Hamiltonian (e.g. the Coulomb interaction for $j = 1$) to obtain an invariant S-matrix [5].

In this paper, we obtain Weinberg's results from a different point of view. This procedure provides some additional insight into the basic underlying mathematical structure. It follows

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directly from its properties as a (massless) representation of the Poincaré group that the massless potential field in any dimension must be a field with gauge degrees of freedom. Moreover, we show that the existence and the admissable form of the gauge transformations follows from the requirement that the field, as a representation of a group which has a solvable factor, must have an equivalence class that corresponds exactly to the property of gauge invariance†. *This new theorem enables us to analyse the gauge properties of a* massless field in any dimension, yielding the explicit form of the gauge transformation and also (without use of the wave equation) the structure of the Hilbert space for the second quantized field theory. As examples for the use of our theorem, we investigate the problem of massless tensor and spinor fields for several cases, and discuss, in particular, the gauge freedom of the massless vector field in any dimension and the possibility of a gravitational, gauge-invariant conformal field for $d = 6$. For a spinor field, we find that there are two possibilities, i.e. spinor fields may be gauge fields, or may have a restricted handedness and then violate parity. We furthermore prove that the harmonic coordinate condition for the linearized gravitational field theory, which appears in the standard literature as a consequence of the freedom to make a *general* coordinate transformation, in fact follows, as a condition for the existence of a massless spin-2 tensor field corresponding to weak gravitational radiation on a flat background spacetime.

2. Vector fields in general dimension

Let $A(x)$ be some field in an arbitrary flat manifold, a generalized spacetime, with metric g of signature (p, q) (where we consider q to be the 'time-like' indices), and suppose that it has some definite transformation properties, i.e. as a spinor or tensor field. Let us consider the Fourier transform $A(k)$. We define $A(k)$ as a massless field if the field has support for k such that $k^\alpha k_\alpha = 0$. Let $\Lambda^\alpha{}_\beta$ be an element of the homogeneous matrix-valued isometry group Λ of the manifold, i.e. $\Lambda = \mathrm{O}(p, q)$ with the generators $L_{\mu\nu} = -L_{\nu\mu}$ $(\mu, \nu = 0, \ldots, d-1)$ satisfying the commutation relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho}) .$$

We denote by $\tilde{\Lambda}^\alpha{}_\beta$ an element of the subgroup of $\tilde{\Lambda}(k)$ that stabilizes k^α, i.e.

$$\tilde{\Lambda}^\alpha{}_\beta k^\beta = k^\alpha \tag{1}$$

where $\tilde{\Lambda}(k)$ is the little group (or stability group) of k. We discuss the structure of this little group below. Under the action of the little group

$$A'(k') = A'(\tilde{\Lambda}k) = A'(k) . \tag{2}$$

If A is a scalar field, then it is invariant under this subgroup of transformations, but if A has indices with spinor or tensor transformation properties then $A' \neq A$. We shall define the components which span an irreducible representation of the (universal covering group of the) little group $\tilde{\Lambda}(k)$ as the polarizations of the field. They are the physical spacetime degrees of freedom of the field. In the case of massless fields, one can show that the

† This result does not, of course, select a particular choice of gauge, such as Lorentz, or Coulomb, but proves that a massless tensor or spinor field must have what we recognize as gauge degrees of freedom.

little group is a semidirect product of a maximally connected semisimple subgroup and a 'translation' part which is a maximally simply connected solvable normal subgroup. There is a general theorem that in such a case the 'translation' subgroup must be represented trivially in any finite-dimensional irreducible representation of the group [6]. In this paper we use this property to find useful information about the field in a simple and general way.

In the four-dimensional Minkowski manifold, the little group of a massless field is isomorphic to $E(2)$, the isometry group of the Euclidian plane (for example, [6]). Let us take, without loss of generality, k to be in the $\hat{z}$ direction. Then this group has three generators of the form [4]

$$J_3 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_1 = \mathrm{i}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$

and the algebra is

$$[J_3, L_1] = \mathrm{i}L_2 \qquad [J_3, L_2] = -\mathrm{i}L_1 \qquad [L_1, L_2] = 0\,. \tag{4}$$

L_1 and L_2 generate the 'translation' subgroup. This group is solvable and admits only one-dimensional (or infinite-dimensional) representations so that A^μ (finite-dimensional) must be stable under the action of the translation part. Since the translation generators are nilpotent, the finite translations are represented as

$$T_1(\alpha) = \mathrm{e}^{-\mathrm{i}\alpha L_1} = \begin{pmatrix} 1+\alpha^2/2 & \alpha & 0 & -\alpha^2/2 \\ \alpha & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \alpha^2/2 & \alpha & 0 & 1-\alpha^2/2 \end{pmatrix} \tag{5}$$

$$T_2(\beta) = \mathrm{e}^{-\mathrm{i}\beta L_2} = \begin{pmatrix} 1+\beta^2/2 & 0 & \beta & -\beta^2/2 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & -\beta \\ \beta^2/2 & 0 & \beta & 1-\beta^2/2 \end{pmatrix}. \tag{6}$$

The field A^μ transforms under the translational part as

$$A'^\mu(k) = \begin{pmatrix} A^t + \eta\Delta + \alpha A^x + \beta A^y \\ A^x + \alpha\Delta \\ A^y + \beta\Delta \\ A^z + \eta\Delta + \alpha A^x + \beta A^y \end{pmatrix} \tag{7}$$

where $\eta \equiv (\alpha^2 + \beta^2)/2$ and $\Delta \equiv A^t - A^z$. If A^μ is required to be literally stable under this transformation, one finds that $\Delta = A^x = A^y = 0$. It then follows that the only excitations one can create are of the form

$$([a_t - a_z]^\dagger)^n|0\rangle\,.$$

Under the assumption of covariant commutation relations for the creation and annihilation operators,

$$[a^\mu(k), a_\nu^\dagger(k')] = \delta^\mu{}_\nu \delta(k - k') \tag{8}$$

these states have zero norm. Such a field, i.e. with $A^x = A^y = 0$ and $A^t = A^z$, could exist classically but not quantum mechanically. In order to obtain a non-trivial field which satisfies these conditions, we must admit an equivalence relation in (part of the) vector field, i.e. to assume that A^j is equivalent to A^j plus an additive term of a given type. If we attempt to find a solution with $\Delta \neq 0$ we find that all the field components are defined only up to an equivalence relation, i.e. the fields are trivial. For a non-trivial solution, one must then take $\Delta = 0$, and $A^x, A^y \neq 0$ (one cannot take just A^x or just A^y as physical fields because J_3 mixes them)†. We find that

$$A^t(k) \sim A^t(k) + \alpha A^x(k) + \beta A^y(k) \tag{9}$$

$$A^z(k) \sim A^z(k) + \alpha A^x(k) + \beta A^y(k)\,. \tag{10}$$

This freedom corresponds to the gauge transformation

$$A^\mu(k) \longrightarrow A^\mu(k) + \mathrm{i}k^\mu \Lambda(k) \tag{11}$$

where we identify $\mathrm{i}k^0\Lambda(k)$ and $\mathrm{i}k^z\Lambda(k)$ as $\alpha A^x(k) + \beta A^y(k)$. The Fourier transform of (11) is

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \Lambda(x) \tag{12}$$

which is the well known gauge freedom of the four-dimensional electromagnetic field. We conclude that *the existence of non-trivial physical degrees of freedom for a massless vector field in* $3+1$ *dimensions implies that the field must be a gauge field.* Moreover, one identifies in this way the nature of the gauge group and the form in which these physical degrees of freedom are represented by the field.

The structure of the quantum-mechanical Hilbert space is determined by the physical degrees of freedom (i.e. the polarization states); we discuss this structure in the following.

The conditions $\Delta = 0$ or $A^t = A^z$ cannot be satisfied (quantum mechanically) as an operator identity [7]; instead, one imposes the subsidiary condition by restricting the space of the states by some linear condition, such as the Gupta–Bleuler condition:

$$(a^t - a^z)|\nu\rangle = 0 \tag{13}$$

for any physical state $|\nu\rangle$. The space of the states, therefore, contains a physical, positive-definite norm subspace $\{|\nu_{\mathrm{phys}}\rangle\}$ created by $a^{\dagger x}$ and $a^{\dagger y}$ (or by $(1/\sqrt{2})(a^{\dagger x} \pm a^{\dagger y})$, which creates ± 1 helicity eigenstates), which we call the physical Hilbert space, and a 'ghost' sector that contain at least 1 'ghost' created by $a^{\dagger t} + a^{\dagger z}$. The ghost sector is a zero-norm subspace orthogonal to the physical one. The physical Hilbert space can be defined

† Classically, $\Delta = 0$ means the transversality of the field, $k_\mu A^\mu = 0$. This condition is necessary for the existence of non-trivial field strengths; it does not correspond to a gauge condition. As we shall see, it is an expression of the Gupta–Bleuler condition for the consistent quantization of a gauge field, and, although we have chosen the $\hat{z}$ direction as the direction of k in the Fourier rerpresentation, the condition is completely covariant.

covariantly as the closure of the quotient space $\{|\nu_{\rm phys}\rangle\}/\{|\nu_{\rm ghost}\}$ and the S matrix is unitary with respect to this physical subspace if it is norm-preserving in the whole Hilbert space and the subspace $|\nu_{\rm phys}\rangle$ is invariant under its action [7].

In general dimension d with signature (p, q), it is easy to see that the little group of the massless field with null k is $E(p-1, q-1)$, the Euclidian isometry group of the $(d-2)$-dimensional space with signature $(p-1, q-1)$. Let us take the zero-length vector in a form in which there is just one time-like and one space-like component, which we shall call longitudinal components. In the construction of the matrices of the stability subgroup, one sees that if (without loss of generality) $k_\mu = (k, 0, 0, \ldots, k)$, in addition to the semisimple $O(p-1, q-1)$ part, which is spanned by $L_{i,j}(i, j = 1, \ldots, d-1)$, there are elements that connect the non-zero components of the zero-length vector. These form a commuting set of $d-2$ boosts and rotation matrices which is spanned by $L_i = L_{i0} + L_{id}$ and play the role of the $T(d-2)$ 'translations' group, the maximally solvable subgroup of $E(p-1, q-1)$.

In a general representation of the massless vector k, we denote by $A^{\rm lt}$ the 'longitudinal' component of the time-like part of the field ($A^{\rm lt}$ parallel to k in the 'time' part of the space with no 'space' components) and by $A^{\rm ls}$ the longitudinal component of the space-like part. These components must satisfy $\Delta = 0$ where $\Delta \equiv A^{\rm lt} - A^{\rm ls}$. The action of the gauge (equivalence) transformation is again of the form

$$A^\mu(k) \longrightarrow A^\mu(k) + {\rm i}k^\mu \Lambda(k) \tag{14}$$

where the Fourier transform is

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \Lambda(x) \tag{15}$$

and the Hilbert space has the same structure as in the $O(3, 1)$ case. For the case $q > 1$ the semisimple subgroup is non-compact and the finite-dimensional representations are non-unitary. This case has been treated in [8]. It is interesting to note that if the isometry group is $O(1, 1)$, there is no little group (excluding the conformal symmetry for the massless particles) for massless and massive particles so there are no intrinsic differences between them from the point of view that we have discussed above. In particular, there is no gauge symmetry (as for the Schwinger model [7]).

3. The massless spinor field

Consider now a massless spinor field in chiral representation in $3+1$ dimensions. The transformation property of a Weyl spinor under the Lorentz group is given by

$$\phi_{\rm L} \longrightarrow {\rm e}^{({\rm i}/2)\sigma\cdot[\theta\hat{n}+{\rm i}\varphi\hat{m}]}\phi_{\rm L} \tag{16}$$

$$\phi_{\rm R} \longrightarrow {\rm e}^{({\rm i}/2)\sigma\cdot[\theta\hat{n}-{\rm i}\varphi\hat{m}]}\phi_{\rm R} \tag{17}$$

where $\hat{m}$ is a unit 3-vector in the boost direction and $\hat{n}$ is the direction of the rotation axis. The parameters θ and φ correspond, respectively, to the rotation and the boost. The generators of the translation part of $E(2)$ (when the momentum k is directed along the $\hat{z}$ axis) are L_1 and L_2 as defined above. The new spinors are, under the action of L_1,

$$\phi'_{\rm L} = {\rm e}^{({\rm i}/2)\alpha(-\sigma_2+{\rm i}\sigma_1)}\phi_{\rm L} = \left(1 - \alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\phi_{\rm L} \tag{18}$$

$$\Rightarrow \begin{pmatrix} \phi^1_{\rm L} \\ \phi^2_{\rm L} \end{pmatrix} \longrightarrow \begin{pmatrix} \phi^1_{\rm L} - \alpha\phi^2_{\rm L} \\ \phi^2_{\rm L} \end{pmatrix}$$

$$\phi_R' = e^{(i/2)\alpha(-\sigma_2 - i\sigma_1)}\phi_R = \left(1 + \alpha\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)\phi_R \tag{19}$$
$$\Rightarrow \begin{pmatrix}\phi_R^1\\ \phi_R^2\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_R^1\\ \phi_R^2 + \alpha\phi_R^1\end{pmatrix}$$

and under L_2,

$$\phi_L' = e^{(i/2)\beta(\sigma_1 + i\sigma_2)}\phi_L = \left(1 + i\beta\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\phi_L \tag{20}$$
$$\Rightarrow \begin{pmatrix}\phi_L^1\\ \phi_L^2\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_L^1 + i\beta\phi_L^2\\ \phi_L^2\end{pmatrix}$$
$$\phi_R' = e^{(i/2)\beta(i\sigma_1 - i\sigma_2)}\phi_R = \left(1 + i\beta\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)\phi_R \tag{21}$$
$$\Rightarrow \begin{pmatrix}\phi_R^1\\ \phi_R^2\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_R^1\\ \phi_R^2 + i\beta\phi_R^1\end{pmatrix}.$$

As we see, in this case one can satisfy the triviality representation condition for the $E(2)$ translation part without equivalence relations; such a solution can be achieved by requiring the upper components of the right-hand spinor, as well as the lower components of the left-hand spinor to vanish identically. We know that in the chiral representation, spatial inversion (the parity transformation) is represented by

$$\psi \longrightarrow \gamma^\circ\psi \tag{22}$$

where ψ is the 4-spinor $\begin{pmatrix}\phi_R\\ \phi_L\end{pmatrix}$ and γ° is, in this representation [9],

$$\gamma^\circ = \begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}. \tag{23}$$

Hence the parity transformation interchanges the upper components of the right-handed and left-handed spinors as well as the lower components. Such transformations mix physical components with identically zero components and therefore parity is violated. It is interesting to note that if we allow the freedom of spinor gauge then we can take the lower component of the left-handed spinor and the upper component of the right-handed spinor to be physical fields where the lower right and the upper left will be equivalent to themselves plus *something*. Parity is also violated for this type of field.

4. Massless, gauge invariant tensor fields

Let us now consider the case of a second-rank massless tensor field $T^{\alpha\beta}(k)$. We first ask if there is a massless tensor that is stable under the translation part of the little group without any equivalence relation, i.e. without gauge freedom. The strict stability condition is

$$\tilde{\Lambda}_{tr} T \tilde{\Lambda}_{tr}^T = T \tag{24}$$

where $\tilde{\Lambda}_{\rm tr}$ is any matrix which belongs to the translation part of the little group, and $\tilde{\Lambda}_{\rm tr}^{\rm T}$, its transpose. One finds that the most general tensor in the $O(3,1)$ geometry (when k is in the $\hat{z}$ direction) that satisfies this restriction can be written as a sum of three tensors, an antisymmetric part, a traceless part and a 'conformal' part. The antisymmetric part,

$$F^{\mu\nu} = \begin{pmatrix} 0 & \alpha(k) & \beta(k) & 0 \\ -\alpha(k) & 0 & 0 & -\alpha(k) \\ -\beta(k) & 0 & 0 & -\beta(k) \\ 0 & \alpha(k) & \beta(k) & 0 \end{pmatrix} \tag{25}$$

is exactly of the form of the electromagnetic field tensor for propagating waves, where $\alpha(k)$ and $\beta(k)$ are the linear polarization components of the electric and magnetic fields. The traceless symmetric part,

$$S^{\mu\nu} = \begin{pmatrix} \gamma(k) & 0 & 0 & \gamma(k) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma(k) & 0 & 0 & \gamma(k) \end{pmatrix} \tag{26}$$

is a candidate for the energy–momentum tensor for a massless field with momentum k, it is indeed of the form of the energy–momentum tensor for the electromagnetic field or for gravitational radiation. What we have called the 'conformal' part, has the form

$$G^{\mu\nu} = \begin{pmatrix} -\phi(k) & 0 & 0 & 0 \\ 0 & \phi(k) & 0 & 0 \\ 0 & 0 & \phi(k) & 0 \\ 0 & 0 & 0 & \phi(k) \end{pmatrix} = \phi(k)\eta^{\mu\nu} \tag{27}$$

where $\eta^{\mu\nu}$ is the flat metric diag$(-,+,+,+)$, and $\phi(k)$ satisfies the massless condition. Fields of the form (25) and (26) are, in fact, necessarily massless if they are to maintain their form under Lorentz transformation. They are not in a definite representation of $O(3)$ but they are invariant in form under $E(2)$. On the other hand, a tensor of the form of $G^{\mu\nu}$ can also represent a massive field; a field with this structure is by definition a tensor under the action of the Lorentz group. If one takes $G^{\mu\nu}(x)$ to be a matter field that satisfies, in the massless case, $\partial^\mu\partial_\mu G^{\lambda\nu}(x) = 0$, or in the massive case $\partial^\mu\partial_\mu G^{\lambda\nu}(x) = m^2 G^{\lambda\nu}(x)$, it appears as (at least for free fields) essentially just one non-interacting massless (or massive) Klein–Gordon field. One can, however, consider the conformal field as a representation of the metric itself, i.e. as a general relativistic conformally flat metric. In the case of a conformally flat metric (e.g. of Robertson–Walker type [10]) of the form (we take φ^2 for convenience, in place of the ϕ of (27))

$$g^{\mu\nu} = \varphi^2(x)\eta^{\mu\nu} \tag{28}$$

the scalar curvature in d dimensions is [11]

$$\begin{aligned} R &= \varphi^{-2}\{-2(d-1)\partial^\mu\partial_\mu \ln\varphi(x) - (d-2)(d-1)\partial^\mu \ln\varphi(x)\partial_\mu \ln\varphi(x)\} \\ &= \varphi^{-2}\{-2(d-1)\frac{1}{\varphi}\partial^\mu\partial_\mu\varphi(x) + (d-1)(4-d)\frac{1}{\varphi^2}\partial^\mu\varphi(x)\partial_\mu\varphi(x)\}. \end{aligned} \tag{29}$$

We know from the Einstein equations that if the trace of the energy–momentum tensor vanishes, the scalar curvature R also vanishes. In such a case, when $d = 6$, both terms in (29) have the same coefficient, and it then follows that $\partial^\mu\partial_\mu\varphi^2(x) = 0$, i.e. the metric must be a massless field.

5. Gravitational radiation and the harmonic-coordinates condition

We now turn to the four-dimensional helicity-2 massless field†. It is generally known that the four-dimensional second-rank tensor field contains states with helicity ± 2. On the other hand, the tensor fields we obtained in the previous section are of helicity-zero type (the $S^{\mu\nu}$ and $G^{\mu\nu}$), and a helicity-1 field (the $F^{\mu\nu}$). We now wish to construct a second-rank tensor with physical ± 2 helicity polarization states. Then the Hilbert space must be of gauge type, i.e. it necessarily contains an equivalence relation.

We consider a symmetric, traceless second-rank massless field in four dimensions. Such a tensor has in general 9 independent components of the form

$$T^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{01} & t^{11} & t^{12} & t^{13} \\ t^{02} & t^{12} & t^{22} & t^{23} \\ t^{03} & t^{13} & t^{23} & t^{33} \end{pmatrix} \tag{30}$$

(where the traceless condition is assumed). The ± 2 helicities are the fields

$$e_+ = t^{11} - t^{22} + 2\mathrm{i}t^{12} \qquad e_- = t^{11} - t^{22} - 2\mathrm{i}t^{12}\,. \tag{31}$$

With the definition,

$$\tilde{\Lambda}_{\mathrm{tr}} T \tilde{\Lambda}_{\mathrm{tr}}^{\mathrm{T}} = \tilde{T}' \tag{32}$$

one finds that the stability condition is

$$e'_+ = e_+ \qquad e'_- = e_-\,. \tag{33}$$

This implies the relations

$$t^{13} = t^{01} \qquad t^{23} = t^{02} \qquad t^{33} = 2t^{03} - t^{00} \tag{34}$$

and the traceless condition becomes

$$2(t^{00} - t^{03}) - (t^{11} + t^{22}) = 0\,. \tag{35}$$

In addition to these identities, one finds that the fields t^{00}, t^{03}, t^{01} and t^{02} are non-physical fields and fall into an equivalence class defined by

$$\begin{aligned} t^{00} &\sim t^{00} + (\alpha^2 + \beta^2)(t^{00} - t^{03}) + 2\alpha t^{01} + 2\beta t^{02} + \alpha^2 t^{11} + 2\alpha\beta t^{12} + \beta^2 t^{22} \\ t^{03} &\sim t^{03} + (\alpha^2 + \beta^2)(t^{00} - t^{03}) + 2\alpha t^{01} + 2\beta t^{02} + \alpha^2 t^{11} + 2\alpha\beta t^{12} + \beta^2 t^{22} \\ t^{01} &\sim t^{01} + \alpha(t^{00} - t^{03}) + \alpha t^{11} + \beta t^{12} \\ t^{02} &\sim t^{02} + \beta(t^{00} - t^{03}) + \alpha t^{12} + \beta t^{22}\,. \end{aligned} \tag{36}$$

We see that the gauge freedom corresponding to this equivalence class can be written as

$$T^{\mu\nu}(k) \longrightarrow T^{\mu\nu}(k) + k^\mu \Lambda^\nu(k) + k^\nu \Lambda^\mu(k)$$

† In general, any plane wave ψ which is transformed by a rotation of any angle θ about the direction of propagation into $\psi' = \mathrm{e}^{\mathrm{i}h\theta}\psi$ is said to have helicity h. The propagation is chosen in the z, i.e. 3 direction.

where $\Lambda^{\mu}(k)$ is defined by (36). One sees that $t^{00} - t^{03}$ is gauge invariant and can be a candidate for a physical field, but the assumption of covariant commutation relations

$$[a^{\mu\nu}(k), a^{\dagger}_{\sigma\lambda}(k')] = \delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda}\delta(k - k') \tag{37}$$

(where the $a^{\mu\nu}$s are defined as the annihilation operators of the corresponding tensor field excitations) implies that $t^{00} - t^{03}$ is a zero-norm (ghost) field. It follows that the condition that all the trace expectation values vanish is

$$\langle \nu'_{\rm phys}|t^{11} + t^{22}|\nu_{\rm phys}\rangle = 0\,. \tag{38}$$

At this stage, with the help of (38) and a simple transformation of (30) to helicity-diagonal form, one can see that there are only 2 physical components in the tensor field, i.e. e_+ and e_-. The conditions (34) (satisfied as a Gupta–Bleuler-type restriction on the Hilbert space) are exactly the conditions to eliminate negative-norm states in this theory, as can be seen from the commutation relations. The gauge freedom now becomes

$$\begin{aligned}
t^{00} &\sim t^{00} + (\alpha^2 + \beta^2)(t^{00} - t^{03}) + 2\alpha t^{01} + 2\beta t^{02} + (\alpha^2 - \beta^2)t^{11} + 2\alpha\beta t^{12} \\
t^{03} &\sim t^{03} + (\alpha^2 + \beta^2)(t^{00} - t^{03}) + 2\alpha t^{01} + 2\beta t^{02} + (\alpha^2 - \beta^2)t^{11} + 2\alpha\beta t^{12} \\
t^{01} &\sim t^{01} + \alpha(t^{00} - t^{03}) + \alpha t^{11} + \beta t^{12} \\
t^{02} &\sim t^{02} + \beta(t^{00} - t^{03}) + \alpha t^{12} - \beta t^{11}\,.
\end{aligned} \tag{39}$$

The helicity-zero states of this theory are of three types. The state corresponding to the components $t^{11} + t^{22}$ is a positive-norm physical field eliminated from the Hilbert space by the traceless condition. The state corresponding to $t^{00}+t^{03}$ is a zero-norm gauge-equivalence class ghost. The field $t^{00} - t^{03}$ corresponds to a gauge-invariant ghost. The helicity-1 fields of this theory, $t^{01} \pm \mathrm{i}t^{02}$, are gauge-equivalence class zero-norm ghosts. All the gauge-equivalence class ghosts can be taken to zero by an appropriate Lorentz transformation where $t^{11} + t^{22}$ is eliminated in all Lorentz frames and $t^{00} - t^{03}$ is a gauge-invariant ghost.

We now consider the possibility that $T^{\mu\nu}$ corresponds to the metric tensor of general relativity, $g^{\mu\nu}$. If the gravitational field is weak, the notions that we have used to define spin, helicity and masslessness through the Fourier transform and the tensor transformation properties under the Lorentz subgroup of coordinate transformations, are still valid. It is then a consequence of (34) and (38) that

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{40}$$

(where $\Gamma^{\lambda}_{\mu\nu}$ denote the corresponding Christoffel symbols), i.e. that the coordinates are harmonic. This choice of coordinatization, usually the result of a choice within the framework of general coordinate transformations (analogous, in general relativity, to the choice of a gauge) [12], is imposed by the transformation properties of the tensor under the Lorentz subgroup and the assumption that the field is massless.

6. Summary

In this paper we have studied the polarization properties of a massless local field in any dimension of spacetime. In the Fourier representation, the field is a finite-dimensional irreducible representation of the corresponding $O(p, q)$ (or its universal covering) which leaves a 'light-like' vector invariant. From this group-theoretical requirement, we are able to define the physical polarizations, and obtain the structure of the Hilbert space representing the states of such a field. We may further deduce that these fields must have equivalence classes corresponding to gauge symmetry.

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