Stochastic Desertification

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Abstract – The process of desertification is usually modeled as a first order transition, where a change of an external parameter (e.g. precipitation) leads to a catastrophic bifurcation followed by an ecological regime shift. However, vegetation elements like shrubs and trees undergo a stochastic birth-death process with an absorbing state; such a process supports a second order continuous transition with no hysteresis. Here we study a minimal model of a first order transition with an absorbing state. When the external parameter varies adiabatically the transition is indeed continuous, and we present some empirical evidence that support this scenario. The front velocity renormalizes to zero at the extinction transition, leaving a finite "quantum" region where domain walls are stable and the desertification takes place via accumulation of local extinctions. A catastrophic regime shift may occur as a dynamical hysteresis, if the pace of environmental variations is too fast.

introduction. – The catastrophic bifurcation and its statistical mechanics analog, the first order transition, play a central role in the physical sciences. In these processes a tiny change in the value of an external parameter leads to a sudden jump of the system from one phase to another. This change is irreversible and accompanied by hysteresis: once the system relaxed to its new phase, it will not recover even when the external parameters are restored.

The relevance of these processes to the ecology of population and communities has been established while ago [1]. Recently, there is a growing concern about the possible occurrence of regime shifts in ecological systems [2–5]. The anthropogenic changes of local and global environmental parameters, from habitat fragmentation to the increasing levels of CO2 in the atmosphere, raise anxiety about the possibility of an abrupt and irreversible catastrophe that may be destructive to the functions and the stability of ecosystems [6]. This concern triggered an intensive search for empirical evidence that may allow to identify an impending tipping point, where the most popular indicator is the phenomenon of critical slowing down [5, 7–10]. Other early warning tools, especially for sessile species, have to do with spatial patterns and the level of aggregation [4, 11, 12]

Of particular importance is the process of desertification, which is considered as an irreversible shift from an "active" vegetation to an "inactive" bare soil state, resulting from an increased pressure (e.g., overgrazing, declines in precipitation). As drylands cover about 41% of Earth land surface, desertification affects about 250 million people around the world [13]. Various models show that, when the vegetation state has a *positive feedback*, like an increased runoff interception or reduced evaporation close to vegetation patches, the system supports two attractive fixed points (alternate steady states) [12, 14]. The bare soil fixed point is stable, since the desert is robust against small perturbation (a small amount of vegetation) for which the positive feedback is too weak, while the active state is self-sustained. Accordingly, a system may cross over from vegetation to bare soil in two routes: First, a disturbance that pushes the system to the basin of attraction of the bare soil fixed point, and second, when the vegetation fixed point losses stability, i.e., when a change of an external parameter takes the system over its tipping point [15]. These scenarios, tipping point catastrophe and the disturbance, have an analog in the dynamics of equilibrium first order transitions, as they correspond to spinodal decomposition and nucleation.

One feature of the system that received only a little attention in the literature is the fact that the bare soil is an *absorbing* (fluctuation free) state. This property has no analog in equilibrium thermodynamics, where every state allows for fluctuations at finite temperature. A trivial implication of this feature is that a finite system is always extinction prone, as it gets stuck in the absorbing state for good once it becomes empty, although the time to extinction may be long [16]. In an infinite system, on the other hand, the active state may acquire stability if the colonization rate of empty sites is faster than the rate of local extinctions.

Here we would like to discuss a spatially explicit model of desertification with *demographic stochasticity*, where the bare-soil is an absorbing state. It turns out that the phase transition becomes continuous, and a new, "quantum" region appears, in which the front velocity renormalizes to zero on a finite domain of system's parameters. In this region the transition takes place via the accumulation of local extinctions. The relevance of this theory to the practical analysis of ecological regime shifts depends on the strength of noise vs. the sweep rate of the external parameter, and we show some empirical evidence that suggest a continues transition scenario. Finally, we will discuss the type of early warning signals one would like to implement in such a transition.

Desert as an absorbing state - a minimal model. – To be specific, let us consider a popular minimal model for desertification, which is a simple version of the Ginzburg-Landau equation. Denoting the biomass density by b, the following PDE allows, for any set of parameters, a (stable or unstable) bare soil $b = 0$ solution,

$$\frac{\partial b}{\partial t} = D\nabla^2 b - \alpha b + \beta b^2 - \gamma b^3. \qquad (1)$$

Here D is the diffusion constant, The control parameter α represent the effect of the (changing) environment, β is a positive constant that represents local facilitation, and the positive constant γ accounts for the finite carrying capacity of the system. When the environment is hostile [positive value of "stress parameter" α, in Eq. (1)] the bare soil (desert) state $b = 0$ is locally stable but local facilitation may allow the system to have another stable state at a finite vegetation density. Negative values of α correspond to better environmental conditions, where the bare soil is unstable (See the bifurcation diagram (lines) in Figure 1).

The deterministic equation (1) admits one or two homogenous solutions, depending on the value of α. Catastrophic desertification occurs beyond the tipping point, i.e., when $\alpha \geq \beta^2/(4\gamma)$, where the system collapses to its desert state following a saddle-node bifurcation. To recover vegetation, the stress parameter α should cross zero (transcritical bifurcation), so the regime shift is irreversible.

When the initial conditions are inhomogeneous, the desert invades the vegetation to the right of the Maxwell (melting) point (MP) $\alpha_m > 2\beta^2/(9\gamma)$, and vegetation invades on its left side (see Fig. 1). Accordingly, when the system is exposed to local disturbances that may form confined spatial domains of the alternative stable state, there is no sudden collapse at the tipping point. Instead, one expects a "gradual global transition" at the Maxwell point, where any local disturbance which is large enough to reach the basin of attraction the alternate state yields a propagating front that spreads until the invading phase takes over the system. This phenomenon was emphasized recently by Bel et. al. [17] who stressed that, in such a scenario, signals that indicate the proximity of the tipping point, like critical slowing down, are not relevant anymore.

The difference between theories that emphasize the tipping point and the approach of [17] has to do with the rate of variation of the external parameters. This distinction is analogous to the situation in equilibrium first-order transitions. When the external parameter (temperature, magnetic field) is varied rapidly, the transition occurs abruptly at the spinodal point, where the metastable phase becomes unstable. Under slow sweep, thermal fluctuations are always strong enough to produce alternate phase nuclei that may expand and take over the system, so the transition takes place at the melting point.

However, stochasticity in an ecosystem occurs even when rates of demographic processes (birth, death, migration etc.) are independent of time, reflecting the randomness of the birth/death process at the individual level [4,11,18,19]. For example, if B represent a unit of biomass (a shrub, say), the quadratic term of Eq (1) may emerge as the deterministic limit of the process $B+B \xrightarrow{\beta} 3B$, the cubic term emerges from $B+B+B \xrightarrow{\gamma} \oslash$ and the linear term corresponds to $B \xrightarrow{\alpha} \oslash$ (if $\alpha \geq 0$) or $B \xrightarrow{\alpha} 2B$ if $\alpha \geq 0$. *Demographic stochasticity* of this kind yields, for a population of size N, fluctuations amplitude that scale with $\sqrt{N}$. This demographic noise may be the driver of a local disturbance that leads to the fixation of the invading phase, a process that was analyzed recently in [16].

As mentioned above, demographic stochasticity has another aspect, that have no analog in equilibrium thermodynamics. Once the discretization of agents is taken into account, the desert state becomes absorbing and is not affected by fluctuations. The phase of a spatial system depends on the ratio between the rate of these stochasticity-induced local extinctions and the recruitment of bare-soil patches by neighboring vegetation. The result is a second-order extinction transition (ET), in which the vegetation density decreases continuously until it reaches zero at the transition point [20]. As shown by Kockelkoren and Chaté [21], this transition belongs to the directed percolation universality class [22].

In figure 1 we show the steady state density of vegetation for a 1d stochastic model with different values of α and for different strength of the demographic noise, together with the deterministic bifurcation diagram. Our simulation technique is close to the split-step method used by [21, 23, 24]: an Euler integration of Eq. 1 (with $\Delta t = 0.001$, $1d$ lattice of $L = 10000$ sites, asynchronous update) is interrupted every ζ (note that large values of ζ corresponds to weak demographic noise and vice versa)

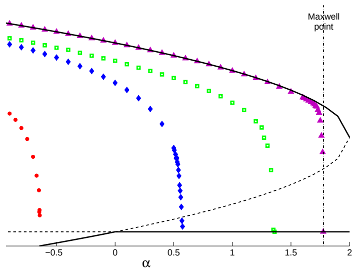

Fig. 1: **The desertification transition**. The lines represent the possible steady states of the spatially homogenous solution of Eq. (1) with $\beta = 0.4, \gamma = 0.02$. Full lines correspond to stable fixed points, dashed lines to unstable points. The transcritical bifurcation at $\alpha = 0$ and the saddle-node bifurcation (tipping point) at $\alpha = 2$ are clearly seen. The Dash-dot line indicates the Maxwell point. The symbols are the steady state density obtained from numerical solutions of the process with different αs for $D = 0.2, \zeta = 30$ (circle), $D = 0.2, \zeta = 60$ (diamond), $D = 10, \zeta = 30$ (Square), $D = 0.2, \zeta = 3000$ (Triangle). The transition point cannot cross the Maxwell point.

generations when the value of b_i at every site i is replaced by an integer, taken from a Poisson distribution with an average b_i.

Of special interest are three phenomena demonstrated in Figure 1:

- The transition is indeed continuous, and we have verified that it belongs to the DP equivalence class, by measuring the critical exponents at the transition (results not shown).

- The ET point is always to the left of the Maxwell point. This feature is also expected, since any noise allows eventually for a large local "hole" and to the right of the MP this disturbance spreads [17], so the steady state must be empty.

- Figure 1 also indicates that, at least when the noise is relatively weak, the vegetation steady state density decays *linearly* as the system approaches the extinction transition. This feature is not trivial: the DP theory predicts a steady state density that scales like $\Delta^{\tilde{\beta}}$, where Δ is the distance from the transition and $\tilde{\beta} = 0.27$ in 1d. We have verified that this is indeed the case very close to the transition point (result not shown here). However, as the system should converge to the deterministic limit at large Δ, the transition region is very narrow (for a general analysis of the transition zone problem, see [25]) and the decay of the steady-state density appears to be linear almost all the way down to $b = 0$.

The hypothesis of a second-order, reversible desertification transition with a linear decay of the steady-state density in the transition regime, is supported by two pieces of data. Reversibility is suggested by a few recent studies, showing a recovery from desertification when the external pressure (grazing, in most cases) has been removed [26–30]. Some evidence for linearity are suggested in Figure 2, where the desertification process of the Sahara during the mid-Holocene is traced through the eolian dust record of Site 658C [31]. The flux of terrigenous sediments seem to grow linearly during the transition period. If this flux reflects the steady state density of an adiabatic environmental change, this linear dependence is in agreement with the predictions of our model.

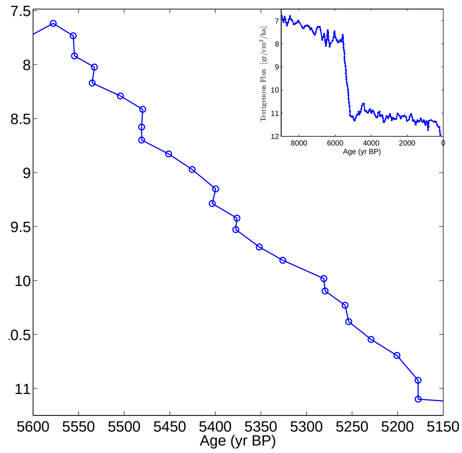

Fig. 2: The mid-Holocene desertification of the Sahara, as expressed by the increase of the flux of terrigenous dust, during the last 9000 years (inset) and during the transition period (main panel, modified from [31]). The transition is assumed to be triggered by a gradual and weak decline of the Northern Hemisphere summer insolation [2, 31].

Note that the Sahara desertification data are usually interpreted (see, e.g., [2]) as an evidence for a catastrophic, first order transition, since the growth of terrigenous sediments *percentage* through time appears to be exponential. However, as stressed in [32], the use of component percentages in marine sediments can be misleading, because the total sediment must always sum to 100%. The long timescales involved (about 500 years) also suggest an alternative mechanism.

The quantum regime. – What happens in the parameter region that lays between the extinction and the Maxwell point? On the one hand if the desert state is not absorbing, vegetation invades the bare soil for these parameters. On the other hand, with absorbing state taking into account the system is beyond the extinction transition. How desertification occurs? To address this issue we have studied the system with inhomogeneous initial conditions and monitoring the growth of the overall density vs. time we have measured the front velocity v.

In the deterministic limit one expects a finite velocity that vanishes (and changes sign) only at the MP. Indeed,

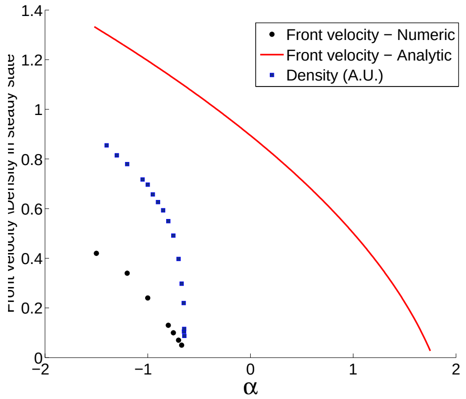

Fig. 3: **Invasion velocity renormalization.** Front velocity is shown against α. Squares represent the steady state density of the stochastic simulation for the same value of α (in arbitrary units). Circles represent the front velocity measured in the simulation. Parameters are $\beta = 0.4, \gamma = 0.02, dt = 0.01, D = 0.2, \zeta = 30$. Given the numerical inaccuracies close to the transition, these two sets of data seem to reach zero at the same point. The solid line correspond to the analytic expression 2. (for these parameters $\alpha_{MP} = 1.778$). Front velocity was measured by monitoring the linear growth rate of the b density. The initial conditions are vegetation for $5000 < x < 10000$ and bare soil for $1 < x < 5000$.

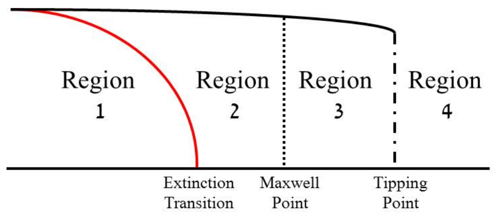

Fig. 4: **Modes of desertification - a schematic cartoon.** The steady-state density of vegetation (red line) approaches zero at the extinction transition. In the "quantum" regime between this point and the Maxwell point (region 2) the desertification happens in a series of local collapses. In region 3 the desert invades vegetation, and the local collapses are superimposed on front propagation. In region 4 the collapse is global, and the vegetation decays uniformly and exponentially - this is the scenario of catastrophic regime shift.

the velocity satisfies,

$$v = \pm\sqrt{2D}\left(\frac{-\alpha}{m} + \frac{m}{2}\right) \qquad (2)$$

where

$$m \equiv \sqrt{-\alpha + \frac{\beta^2}{\gamma}\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\gamma\alpha}{\beta^2}}\right)}. \qquad (3)$$

Note that the front change also its characteristic, from a Ginzburg-Landau front to Fisher type II, at the transcritical bifurcation [33,34]. Stochasticity *without* an absorbing state may modify the location of the MP, but the general structure remains the same [35]. In our case, where the system admits an absorbing state, the situation is different. As shown in Figure 3, under demographic stochasticity the velocity renormalizes to zero at the extinction transition point, so there is a whole parameter region where the velocity vanishes.

The emerging insights are summarized in Fig 4. For every set of parameters (diffusion, noise, nonlinear interaction) the system will be found in one out of four different phases. Above the extinction transition (region 1) vegetation saturates to an equilibrium value and will invade a nearby bare-soil region. The steady state density vanishes at the extinction transition, but desertification may take place in different modes. In the "quantum" region 2 (between the ET and the MP) the desert does not invade, and the transition comes about by accumulation of local extinctions eventuating a global collapse. In region 3 these collapses are accompanied by the desert invasion as predicted by [17] and the dominant effect depends on the size of the system, the rate of local extinctions and the velocity of the front. Finally, beyond the tipping point (Region 4) the deterministic active fixed point loses its stability and vegetation collapse exponentially, simultaneously all over the place.

All in all, when demographic noise and the absorbing state are taken into account, one finds that if environmental changes (like the rate of variations of α) are adiabatic, the phase transition is a continuous, second order one, without hysteresis. The catastrophe scenario - a global collapse after the crossing of the tipping point, followed by an irreversible transition between alternative stable state, can never be realized if the sweep of the external parameter is infinitely slow. As long as $\zeta < \infty$ the transition is second order and, even more importantly, it cannot take place beyond the Maxwell point, so the tipping point is completely disparate from the extinction transition. This is also the case for theories with local disturbances, even without an absorbing state [17], but in that case the transition is discontinues (involves an order parameter jump) and sticks to the MP, while the quantum transition occurs before it. Accordingly (as already pointed out [17]), the attempts to identify an impending catastrophe by analyzing fluctuation dynamics, utilizing the critical slowing down as an early warning signal, appears to be useless.

The studies of tipping points and early warning signals may be relevant to the desertification problem only if the environmental change is non-adiabatic, where the irreversibility has to be interpreted as a dynamical hysteresis [36]. This behavior is demonstrated in Figure 5. Dynamical hysteresis is unavoidable close to the extinction transition when the response of the system becomes slower than the pace of environmental change, but its effect may be very weak.

As each of the regions 1-4 (in Fig. 4) has its own characteristic timescale, the conditions for a "rapid" sweep rate are different in different regions. The deterministic picture is relevant only when the sweep rate for α is faster than any other process in the system. However, in such a case the implementation of critical slowing down indi-

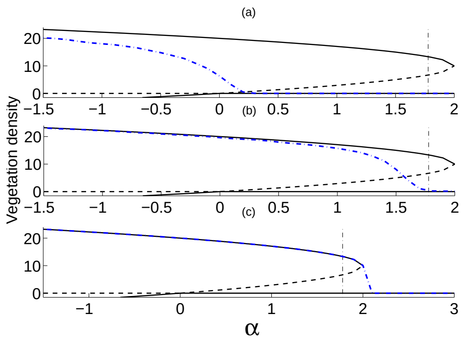

Fig. 5: **Dynamical hysteresis**: Vegetation density (Dash-dot) vs. α, depicted with the deterministic bifurcation diagram as a background (Solid), for $\alpha = -1.5 + s \cdot t$, $s = 10^{-5}$, with $\zeta = 40$ (a) 200 (b) and 1000 (c).

cators close to the tipping point, assuming that one can trace the relaxation of fluctuations before the shift, may also become inefficient.

Apparently, a more reliable early warning indicators may be obtained from the monitoring of voids dynamics. In region 1, the chance of a bare-soil patch to grow is inversely correlated with its size. Region 2 is characterizes by stable domain walls, and the only process that allows for bare-soil cluster to grow is a merge with nearby void. In region 3 and the chance to grow is positively correlated with the cluster size. We intend to pursue these ideas in future studies.

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REFERENCES

[1] May R. M., *Nature*, **269** (1977) 471.

[2] Scheffer M., Carpenter S., Foley J. A., Folke C. and Walker B., *Nature*, **413** (2001) 591.

[3] Rietkerk M., Dekker S. C., de Ruiter P. C. and van de Koppel J., *Science*, **305** (2004) 1926.

[4] Kéfi S., Rietkerk M., Alados C. L., Pueyo Y., Papanastasis V. P., ElAich A. and De Ruiter P. C., *Nature*, **449** (2007) 213.

[5] Scheffer M., Bascompte J., Brock W. A., Brovkin V., Carpenter S. R., Dakos V., Held H., Van Nes E. H., Rietkerk M. and Sugihara G., *Nature*, **461** (2009) 53.

[6] Duraiappah A. K., Naeem S., Agardy T. and Assessment M. E., *Ecosystems and human well-being: biodiversity synthesis* (Island Press Washington DC) 2005.

[7] Dakos V., Scheffer M., van Nes E. H., Brovkin V., Petoukhov V. and Held H., *Proceedings of the National Academy of Sciences*, **105** (2008) 14308.

[8] Boettiger C. and Hastings A., *Proceedings of the Royal Society B: Biological Sciences*, **279** (2012) 4734.

[9] Lenton T. M., *Nature Climate Change*, **1** (2011) 201.

[10] Dai L., Vorselen D., Korolev K. S. and Gore J., *Science*, **336** (2012) 1175.

[11] Scanlon T. M., Caylor K. K., Levin S. A. and Rodriguez-Iturbe I., *Nature*, **449** (2007) 209.

[12] Solé R., *Nature*, **449** (2007) 151.

[13] Reynolds J. F., Smith D. M. S., Lambin E. F., Turner B., Mortimore M., Batterbury S. P., Downing T. E., Dowlatabadi H., Fernández R. J., Herrick J. E. *et al.*, *science*, **316** (2007) 847.

[14] Suding K. N., Gross K. L. and Houseman G. R., *Trends in Ecology & Evolution*, **19** (2004) 46.

[15] Rietkerk M., Ketner P., Stroosnijder L. and Prins H. H., *Journal of Range Management*, (1996) 512.

[16] Meerson B. and Sasorov P. V., *Physical Review E*, **83** (2011) 011129.

[17] Bel G., Hagberg A. and Meron E., *Theoretical Ecology*, **5** (2012) 591.

[18] Manor A. and Shnerb N. M., *Journal of theoretical biology*, **253** (2008) 838.

[19] Realpe-Gomez J., Baudena M., Galla T., McKane A. J. and Rietkerk M., *Ecological Complexity*, **15** (2013) 97.

[20] Behar H., Shnerb N. and Louzoun Y., *Physical Review E*, **86** (2012) 031146.

[21] Kockelkoren J. and Chaté H., *arXiv preprint cond-mat/0208497*, (2002) .

[22] Hinrichsen H., *Advances in physics*, **49** (2000) 815.

[23] Moro E., *Physical Review E*, **70** (2004) 045102.

[24] Bonachela J. A., Muñoz M. A. and Levin S. A., *Journal of Statistical Physics*, **148** (2012) 723.

[25] Kessler D. A. and Shnerb N. M., *Physical Review E*, **85** (2012) 051138.

[26] Fuhlendorf S. D., Briske D. D. and Smeins F. E., *Applied Vegetation Science*, **4** (2001) 177.

[27] Rasmussen K., Fog B. and Madsen J. E., *Global Environmental Change*, **11** (2001) 271.

[28] Valone T. J., Meyer M., Brown J. H. and Chew R. M., *Conservation Biology*, **16** (2002) 995.

[29] Zhang J. Y., Wang Y., Zhao X., Xie G. and Zhang T., *New Zealand Journal of Agricultural Research*, **48** (2005) 277.

[30] Allington G. and Valone T., *Journal of Arid Environments*, **74** (2010) 973 .

[31] deMenocal P., Ortiz J., Guilderson T., Adkins J., Sarnthein M., Baker L. and Yarusinsky M., *Quaternary Science Reviews*, **19** (2000) 347.

[32] Adkins J., deMenocal P. and Eshel G., *Paleoceanography*, **21** (2006) PA1206.

[33] Ben-Jacob E., Brand H., Dee G., Kramer L. and Langer J., *Physica D: Nonlinear Phenomena*, **14** (1985) 348.

[34] Kessler D. A., Ner Z. and Sander L. M., *Physical Review E*, **58** (1998) 107.

[35] Khain E., Lin Y. and Sander L., *EPL (Europhysics Letters)*, **93** (2011) 28001.

[36] Tredicce J. R., Lippi G.-L., Mandel P., Charasse

B., Chevalier A. and Picque B., *American Journal of Physics*, **72** (2004) 799.