

Directed percolation and the Extinction transition on diffusive substrate

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Abstract

The extinction transition on a one dimensional heterogeneous substrate with diffusive correlations is studied. Diffusively correlated heterogeneity is shown to affect the location of the transition point, as the reactants adapt to the fluctuating environment. At the transition point the density decays like $t^{-0.159}$, as in directed percolation. However, the scaling function describing the behavior away from the transition and other critical exponents shows significant deviations from the known DP behavior. It is suggested, thus, that the off-transition behavior of the system is governed by local adaptation to favored regions.

I. INTRODUCTION

The extinction transition in the stochastic birth-death process is very important in many branches of science and serves also as a paradigmatic example of an out of equilibrium phase transition [1–3]. Given a homogeneous substrate and a single absorbing state, Grassberger [4] and Janssen [5] conjectured that the microscopic details of the stochastic process are irrelevant close to the extinction point and the transition is in the directed percolation (DP) universality class. The basic rationale behind this conjecture is that a spatially extended system decomposes, close to the transition, into active and inactive zones, where after each typical period of time there is certain probability for an active state to die, to survive, or to infect its inactive neighbors. If these regions are considered as lattice points on a d dimensional array, the chance of an active site to survive or to infect its neighbors within a unit time is equivalent to the chance that a bond exists between a lattice point at time t and its neighbors in a subsequent replica of the system at $t + 1$. Accordingly, the extinction transition happens when the bond density is exactly at threshold for an infinite cluster in a $d + 1$ dimensional system, and the transition belongs to the directed percolation universality class in $d + 1$ dimensions.

The Grassberger-Janssen conjecture has proven to be extremely robust, and a large number of stochastic models that admit an extinction transition were shown to belong to the DP equivalence class if the substrate is homogenous [1, 2]. It was further shown that spatio-temporal substrate noise (i.e., birth-death rates that fluctuate in space and time with only short range correlations) is an irrelevant perturbation close to the transition, so small noise is averaged out and leaves the DP transition unaffected [1]. Quenched (time independent) disorder, on the other hand, is a relevant perturbation [6] and seems to change the nature of the transition. In that case a Griffiths phase exists between the active and the inactive regions [7]. In the parameter region that corresponds to the Griffiths phase the survival of an active region depends on the local properties of the substrate, *not* on activation by neighboring regions. In particular, for each time scale the survival of active regions depends on the existence of spatial domains that admit high carrying capacity [7, 8]. Although stochastic fluctuations guarantee extinction for any finite sample, the time scale for that grows exponentially with the carrying capacity. This implies that exponentially rare spatial regions with high birth rate support the population for exponentially large times. An

optimization argument shows that in such a case the survival until t is dominated by rare spatial fluctuations of linear size $L \sim \log(t)$; accordingly, the density falls algebraically with time. The Griffiths phase is located between an extinction region, where essentially no good islands exist, and the active phase, where good islands infect each other to yield a never dying process.

What happens, then, if the disorder is neither annealed nor quenched, but is diffusively correlated? Two contradictory arguments may be advanced. In terms of the renormalization group properties of Reggeon field theory, diffusive disorder is a relevant operator, so in principle one may expect that the transition will be in a different equivalence class. This was suggested, in fact, by Kree, Schaub and Schmittmann [9], who then proceeded to develop a perturbative renormalization group based ϵ expansion around four dimensions, a treatment that yield predictions of new (non-DP) critical exponents.

On the other hand, it is clear that the basic intuitive justification for the Grassberger-Janssen conjecture is applicable as long as the favored regions are mobile [10]. Even if the process is confined to the sites where the diffusive catalysts exist, these catalysts move randomly in space, thus again any region may survive, become inactive or infect neighboring regions. One may suggest, accordingly, that even in the presence of diffusive catalysts the extinction transition will still be within the DP equivalence class.

Of particular interest is the applicability of the diffusive substrate scenario to the study of the SIS epidemic model with diffusing agents [11, 12]. In the deterministic limit, where the infection rate is σ and the recovery rate is μ , the rate equations for an SIS epidemic are:

$$\begin{aligned}\dot{S} &= D_S \nabla^2 S - \sigma SI + \mu I \\ \dot{I} &= D_I \nabla^2 I + \sigma SI - \mu I.\end{aligned}\tag{1}$$

For the sake of simplicity let us assume that both infected (I) and susceptible (S) individuals are wandering randomly at the same rate, so the effective diffusion constants are identical. If the total density of agents is $\rho(x) \equiv I(x) + S(x)$ eqs. (1) take the form

$$\begin{aligned}\dot{\rho} &= D \nabla^2 \rho \\ \dot{I} &= D \nabla^2 I + (\sigma \rho - \mu) I - \sigma I^2.\end{aligned}\tag{2}$$

Thus, if ρ is homogenous the extinction transition is in the DP equivalence class, while for heterogenous $\rho(x)$ the extinction transition takes place in the presence of diffusively

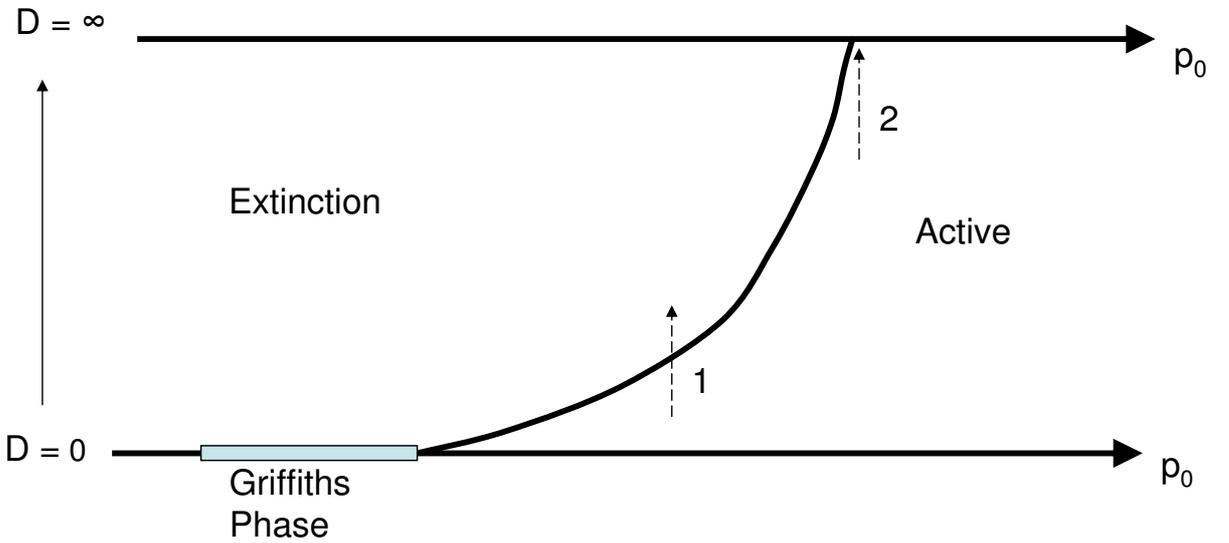
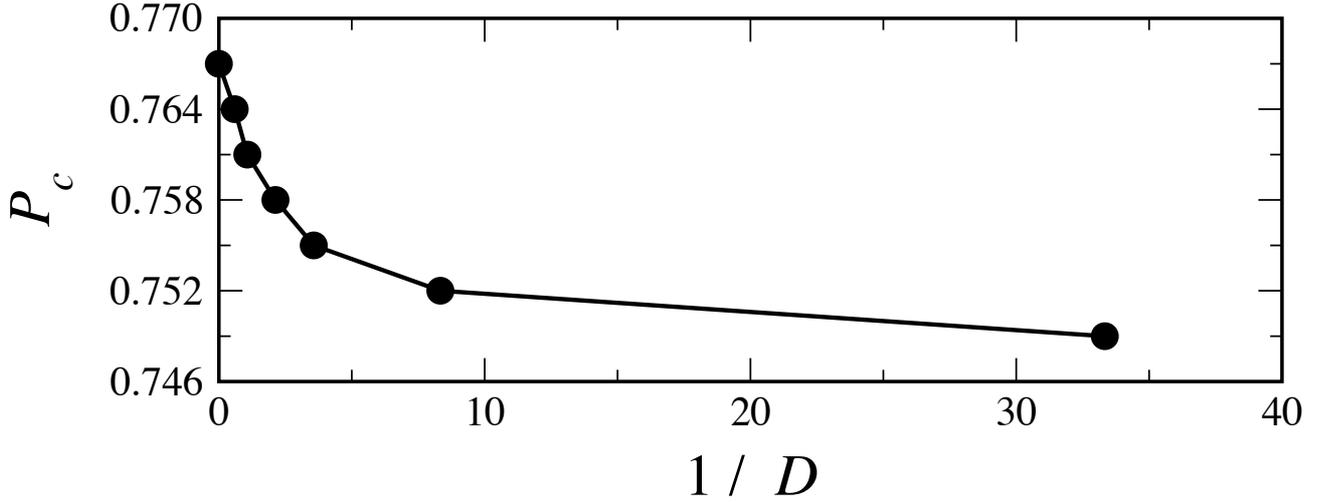


FIG. 1: A sketch of the phase space in the $D - p_0$ plane (lower panel), and the actual results measured for the contact process with a diffusively correlated substrate. As D approaches infinity the environmental stochasticity becomes uncorrelated in space and time and the system approaches the homogenous limit of the DP transition. The critical value of the birth rate, p_c , coincides with the value measures for a contact process on homogenous 1d substrate, as demonstrated in the upper panel. On the other hand as D become smaller the effect of local adaptation is significant and the transition is shifted towards the extinction region. As $D \rightarrow 0$ the transition point appears to coincide with the right edge of the Griffiths phase. In the upper panel the actual values of p_c , the critical values of p_0 , are shown. For the same system parameters the quenched disorder transition (from the Griffiths phase to the active one) takes place at $p_c = 0.746$. Data were obtained from MC simulations of the 1-d contact process with $L = 10^4$ and $\eta = 0.1$.

correlated birth rate.

A few numerical experiments, based on the diffusing SIS model, have reported contradictory results. The authors of [11, 12], using agent-based simulations, reported that the transition in the presence of diffusive substrate belongs to a different universality class than DP. On the other hand, Dornic [13] and coworkers, using an improved Langevin equation simulation technique, have measured the density decay rate at the transition and found the exponent 0.159, as in the DP class. The explanation given [11, 13] to this apparent contradiction is that the Langevin equation fails to reflect the actual nature of an agent-based model.

Here we present numerical results that suggest an alternative answer: it seems that **at** the transition point the critical exponent that characterizes the extinction transition is indeed identical with the DP exponent. However, the subcritical and the supercritical behavior are not described by the directed percolation critical exponents and the system fails to support a scaling function.

The model used here is a contact process (CP) on a fluctuating substrate. It takes place on a 1d lattice with L sites (with periodic boundary conditions), where any lattice site is either occupied by an agent or empty. In an elementary Monte-Carlo reaction step an agent is chosen at random and then attempts to multiply with probability p or "die" (be eliminated from the sample) with probability $1 - p$. If the agent does attempt to multiply, one of the two neighboring sites is chosen at random and becomes occupied if it is currently empty; if the site is already occupied, nothing happens.

To take into account the fluctuating environment, p is taken to be a space-time dependent fluctuating quantity. At $t = 0$, sites are chosen with equal probability to be either "good" ($p_i = p_0 + \eta$) or "bad" ($p_i = p_0 - \eta$) where i is a site index and η is a constant. Subsequently, each elementary step is chosen to be either a reaction step, as described above, or a catalyst diffusion step. In a diffusion step, a randomly picked site switches its p value with one of its nearest neighbors. The probability of a given step to be a reaction step is taken to be $r = \rho/(\rho + D)$, where D is the catalyst diffusion rate, so that the probability to be a diffusion step is $1 - r$. After each elementary MC step the time counter is advanced by $1/L(\rho + D)$.

The phase diagram is shown in Figure 1. For $D \rightarrow \infty$, the space-time disorder becomes uncorrelated and the system belongs to the DP universality class. At this parameter region the transition occurs at $p_c = 0.767$, the critical value for the contact process on a homogenous

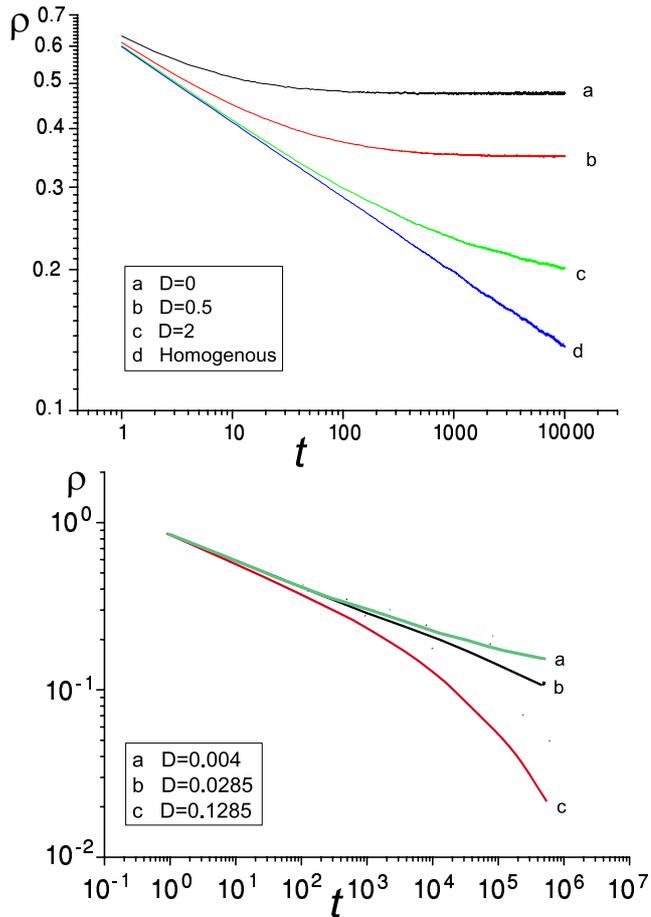


FIG. 2: Density vs. time (logarithmic scale) at the transition and in its vicinity. In the upper panel the dashed line (2) in figure 1 is followed and diffusion is increased with fixed $p_0 = 0.767$ that corresponds to the transition point of the pure contact process. The system is always in its active phase, as suggested by the adaptation argument. In the lower panel the separation line is crossed along arrow (1) of figure 1 ($p_c = 0.749$), and the system crosses from the active to the inactive phase. In both cases, and all other cases checked by us (values correspond to the data points in the upper panel of Fig. 1), the slope at the transition is 0.159, in agreement with the DP theory predictions. Datasets were obtained with lattice size $L = 10^4$ and $\eta = 0.1$

substrate [3, 14]. As D becomes smaller, the catalysts spend more time in certain spatial regions. Agents in these regions produce more offspring in adjacent sites, and when the catalyst jumps to a neighboring site its probability to be occupied is larger than average. This implies that reactants "adapt" to the instantaneous configuration of the catalysts, an effect that yields very strong proliferation in unbounded growth models [15]. Here the growth

is bounded and the effect of adaptation is finite, still the transition point shifts leftward as exemplified in Fig. 1. The location of the extinction transition for the slowest diffusion we were able to measure is very close to the location of the transition from the Griffiths phase at $D = 0$; it seems plausible that the transition line converges to the right end of the Griffiths phase as D approaches zero.

How does the system behaves at, and close to, the transition? According to the standard theory, [2] the DP transition is characterized by a scaling function of the form

$$\rho(t) \sim t^{-\delta} \Phi(\Delta t^{1/\nu_{\parallel}}) \quad (3)$$

Where Φ is a universal scaling function, δ is the universal exponent (in 1d $\delta = 0.159$), Δ is the distance from the transition, $p_c - p_0$, and ν_{\parallel} is the temporal correlation length. This implies that all the curves coincide when plotting ρt^{δ} vs. $\Delta t^{1/\nu_{\parallel}}$. In the presence of diffusive disorder, while the density decreases like $t^{-\delta}$ where δ takes the same numerical value predicted by the DP theory, the scaling function does not exist, as will be shown below.

Figure 2 shows the approach to the transition from the active and from the inactive phase. For any crossing of the transition line checked, either by increasing the diffusion or by changing p_0 , the density at the transition decays like a power law with the DP critical exponent $\delta = 0.159$. However, the off-transition behavior does not obey the DP predictions. As noted above, a graph of $\rho(t)t^{\delta}$ vs. $\Delta t^{1/\nu_{\parallel}}$ should be universal close to the transition. Figure 3 shows that this is, in fact, the case both on a homogenous substrate and for infinite diffusion (i.e., when the locations of the catalysts are uncorrelated in space and time). On the other hand for finite diffusion the long-time behavior fails to fit the universal curve.

These results suggest that the off-transition behavior of the diffusive substrate system differs from that of directed percolation. To validate this, we have checked an off-critical exponents: the growth of ρ_{sat} above the transition (DP theory predicts $\rho_{sat} \sim \Delta^{\beta}$, where in 1d $\beta \approx 0.276$). As demonstrated in Figs. 4, for a diffusive substrate the off-critical exponent differs from that of DP, and β depends on the diffusion constant D ; as $D \rightarrow \infty$ the exponents converge to the value suggested by the DP theory. We have also checked explicitly (data not shown) that $\beta' = \beta$, where β' is the exponent that characterize the mass of the infinite cluster (or the survival probability of a single seed) above the transition [1]. The results for β and δ have been recovered also by simulating the Langevin equation directly, using the technique of [13].

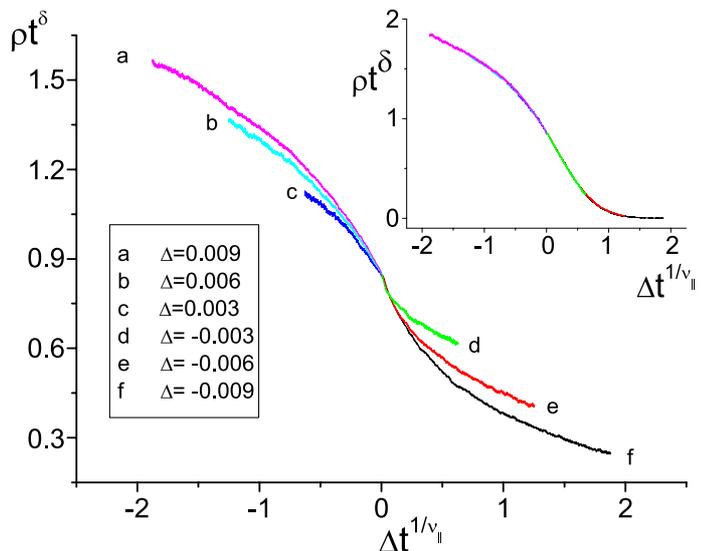


FIG. 3: The universal scaling function for uncorrelated spatio-temporal noise (inset) and for finite diffusion ($D = 0.225$) of the catalysts. Different colors correspond to different distance Δ from the transition, as explained in the legend.

One might think that it is possible to collapse the data shown in Figure 3 using the D dependent β we have just found. As seen in Fig. 3, however, the naive scaling yields a local data collapse close to the transition point at $\Delta = 0$; replacing β with its D dependent value destroys this feature (data not shown). This implies that a one-parameter universal scaling function is not enough to describe the system.

Our numerics to date fails to yield a reliable estimate of the third exponent, the spatial correlation length ξ_{\perp} (DP suggests $\xi_{\perp} \sim \Delta^{-\nu_{\perp}}$ with $\nu_{\perp} \approx 1.097$ in 1d). Qualitatively, however, it is clear that ν_{\perp} is also D dependent and is smaller, for finite D , than the DP predictions. These deviations may be attributed to the *adaptation* of the reactant to the fluctuating environment: the saturation density above the transition is higher than in the homogenous case (thus β is increased), due to the existence of colonies that are localized around the wandering hot spots (thus ν_{\perp} decreased with diffusion).

Two types of noise appear in the problem at hand: the demographic stochasticity associated with the discreteness of individual reactants and the environmental stochasticity associated with the diffusive wandering of the underlying catalysts. For an unbounded growth problem (when the carrying capacity of any spatial point is infinite) this system is equivalent to KPZ growth with correlated disorder, but the KPZ perturbative scheme

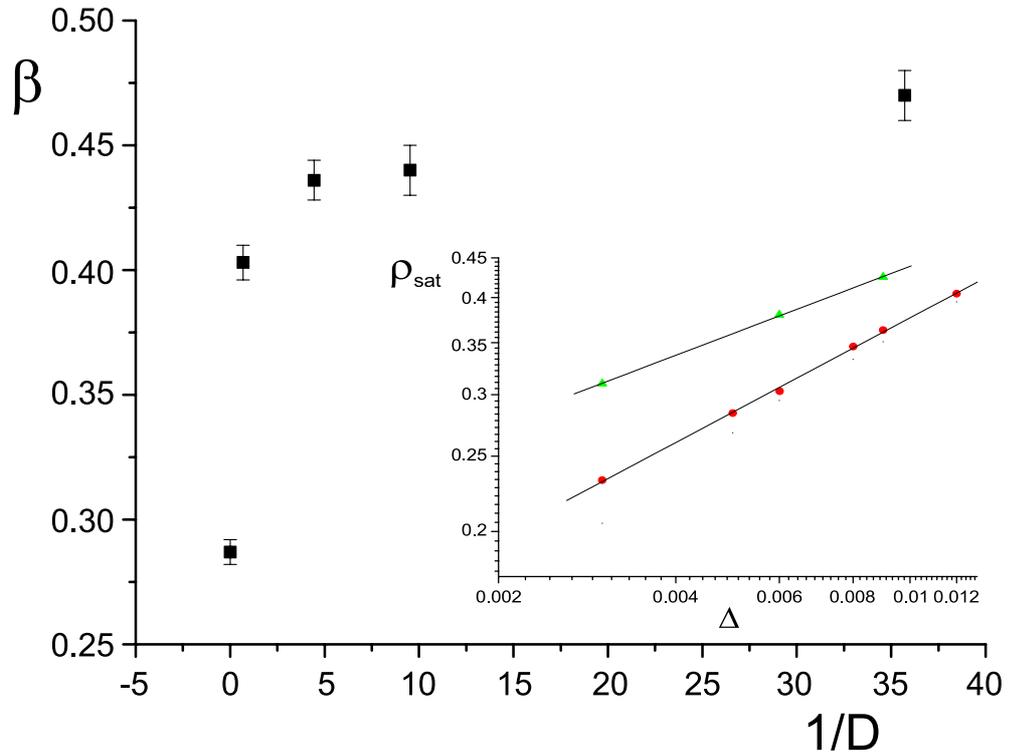


FIG. 4: The exponent β , as measured for different diffusivities, vs. $1/D$. The inset shows examples of the raw data from which the datapoints of the main graph were extracted, from the slopes of least-square fits of $\ln \rho_{sat}$ vs. $\ln \Delta$ (the upper line is the best fit for the infinite D case, the lower one is the lowest diffusion case).

fails for diffusively correlated disorder [17] since local adaptation initiates localized colonies. Indeed, for the unbounded problem extinction happens almost surely (i.e., for any finite spatial domain reactants goes extinct at the long time limit), yet the average population grows faster than exponentially for $d \leq 2$ [16]. This effect disappears in the case of limited growth, as the adapted colonies can no longer grow exponentially forever. Still, it seems that the favored regions dominate the system's behavior above and below the transition. At the transition point survival is based on the ability of catalyst rich zones to "infect" each other, and the microscopic details average out within the diverging correlation length. Once ξ_{\perp} becomes finite different zones are effectively independent and the spatial heterogeneity dictates the local decay; in such a case local adaptation leads to longer survival times and the system behavior resembles the Griffiths phase, thus leading to deviations from the DP universal scaling function.

Recently, Dickman [18] have studied a similar system, contact process with diffusing vacancies, on a one dimensional array of sites, and found different critical exponents for different diffusion levels. In his model, however, vacancies diffuse by exchanging positions with their neighboring sites, so the movement rate of the active agents depends partially on the diffusion of the vacancies. Moreover in the static limit (vacancies are standing still) a one dimensional array is segregated into disconnected regions, so there is no transition point from the Griffith phase to the active state for finite birth rate.

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