

# Nonlocal competition and front propagation in branching-coalescence system.

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The spatial invasion of a stable into an unstable phase is studied for the branching-coalescence process with nonlocal competition. Numerical experiments show that the threshold at the front leading edge, introduced by the discreteness of the reactants, allows for the nonlocal competition to effect the front velocity. However, the front still moves ballistically after a short transient period for any finite range competition length.

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## I. INTRODUCTION

The logistic equation, first presented by Verhulst [1] served for centuries as the fundamental method to describe growth processes with saturation [2], from the growth of bacterial colony on a petri dish [3] and the spread of a favored gene [4] in population to first order phase transitions in the absence of metastable state [5]. Logistic growth on spatial domains, where the motion of the individual reactant is diffusive, was first considered by Fisher [4] and by Kolomogoroff, Petrovsky and Piscounoff [6], and its "mean field" version:

$$\frac{\partial c(x,t)}{\partial t} = D\nabla^2 c(x,t) + ac(x,t) - bc(x,t)^2. \quad (1)$$

It has been shown to support a stable front that propagates in constant velocity  $v_F = 2\sqrt{Da}$ . As this velocity is independent of the saturation term  $b$ , as the front velocity is determined by its leading edge where the nonlinear competition is negligible; the propagation of such "pulled front", thus, depends strongly on the leading edge, where the population density is dilute, rather than on its bulk properties.

As any realistic system is made of discrete reactants (atoms, molecules, animals, cells) it must admit some sort of demographic stochasticity, related to the probabilistic nature of the birth-death-diffusion processes, and to a "threshold", reflecting the fact that the particles density cannot be less than unity. Eq. (1), in that framework, has to be considered as an approximation of an underlying master equation, where averages over higher moments is replaces by powers of the average. For such a pulled front, the corrections to (1) due to discretization are of great importance, since the velocity is determined in the diluted region. The effect of stochastic fluctuations induced by the discrete character of the individual reactants (animals, genes, molecules) was considered either in the strong stochasticity, low density limit [7–9] or in the "semiclassical" limit close to the mean field [10, 11]. It turns out that the basic feature, namely, the ballistic propagation of the front, is robust to discretization, but the velocity of the front is effected by it. In general, as the discretization introduced a "cutoff" along the leading edge, the front velocity lowered with respect to the mean-field predictions.

In many biological situation, e.g., where the invasion of a new species is considered, the offspring and its "parent" (or any other individual) compete for common resource. It is known, for example, that this type of competition decreased the chance of a seed to establish close to an existing plant [12]. This situation may be incorporated into the logistic-diffusive description by the introduction of nonlocal competition term:

$$\frac{\partial c(x,t)}{\partial t} = D\nabla^2 c(x,t) + ac(x,t) - c(x,t) \int_{-\infty}^{\infty} \gamma(x,y)c(y,t)dy, \quad (2)$$

where  $D$  is the diffusion coefficient  $a$  is the growth rate and  $\gamma(x,y)$  is a kernel for the non-local interaction. This modified FKPP equation is the subject of many current studies [13–19], where the main new feature considered is an instability of the homogenous solution that leads to the appearance of ordered [13–15, 17–19] or disordered [13] steady-state spatial patterns on spatial domains.

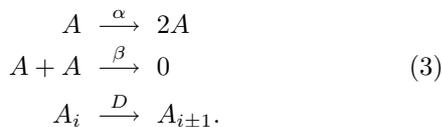
A subject that remains quite unnoticed is the effect of nonlocal competition on the invasion process itself, and in particular on the velocity of the front. Clearly, since the FKPP front is "pulled" one [5] - the effect of the competition on the velocity in the "mean field" framework should be negligible. As long as the competition kernel  $\gamma(x,y)$  in (2) is finite, there is a region on the leading edge where the its effect is negligible and the velocity of this region is still  $v_F = 2\sqrt{Da}$ ; in the continuum approximation the only effect of the competition has to do with the shape of the front, but its velocity remains untouched.

One may suspect, though, that the combined effect of the discreteness of the reactants and the nonlocal competition may alter the invasion characteristics. As explained, discreteness of the reactants implies some sort of "threshold" at the leading edge, thus the effect of nonlocal competition may be large enough at this region. In fact, in recent work by Young and Birch [18] the effect of agents discretization with nonlocal interaction was considered numerically, and one of the results suggest that the front invasion into the unstable phase halts due to the combined effect of strong nonlocal competition and reactant discretization. These findings have motivated this study, where we have tried to consider this effect in detail and to see if, and under what conditions, it is possible to stop the propagation by nonlocal competition. Our study indicates that the front does not really stops,

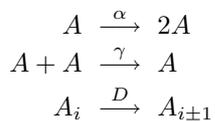
although its velocity is diminished significantly with the interaction strength and range. While discretization does affect the velocity of the Fisher front, the front still propagates linearly as long as the growth rate is larger than zero.

### A. The model

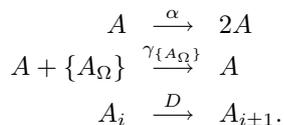
There are several discrete reactants "microscopic" models that yield the FKPP equation as their continuous limit. One of the standard realizations is the branching-annihilation process, that involve the multiplication of an agent ( $A$  is divided into two reactants at rate  $\alpha$ ), and the mutual annihilation of two agents by each other at rate  $\beta$ . All the agents are diffusing with hopping rate  $D$ . Schematically, the branching-annihilation basic ingredients are:



However, this model is inappropriate to our study, as the two basic processes yield an effective "death rate" due to the mutual exclusion of a particle with its offspring. The renormalized model, accordingly, admits an extinction transition at finite  $\alpha/\beta$ , as shown by Grassberger and de la Torre [20] (see also the work of Cardy and Tauber [21]). In order to avoid the extinction transition at finite  $\alpha$ , a branching-coalescence process is used here, where the basic reactions are:



Nonlocal competition is modeled by introducing  $\gamma(r)$ , i.e., by allowing for nonlocal coalescence of agents, and the process elements are:



Where  $\Omega$  is the set of neighbors (particles locating in a neighboring sites) that compete with a specific reactant, and the strength of the competition between two reactant is in general proportional to the distance between them.

Hereon we are using the "top hat" kernel used also in [18], where  $\gamma = \gamma_0$  if  $|x_i - x_j| \leq L$ , where  $x_i$  is the location of the  $i$ -th particle and  $L$  is the interaction range. for  $|x_i - x_j| > L$ ,  $\gamma = 0$ . The "strength" of the nonlocal interaction has to do with both  $\gamma_0$  and  $L$ . The number of particles at a single site is not limited.

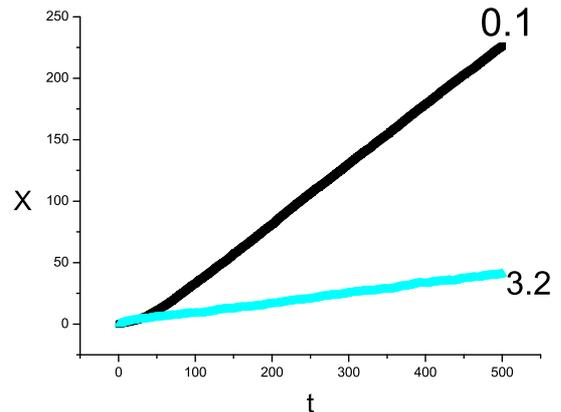


FIG. 1: The location of the rightmost particle in a process described in the text, averaged over 40 different runs. The top hat competition is characterized by  $L = 5$ ,  $D = 0.25$ , and the growth rate  $\alpha = 0.5$ . The results are shown for weak  $\gamma = 0.1$  and strong  $\gamma = 3.2$  competition strength. Clearly the front moves with constant velocity, but the velocity  $v$  is decreasing with  $\gamma$ .

## II. SIMULATION PROCEDURE AND RESULTS

We have run a Monte-Carlo simulation of the branching-coalescence model described above on a one dimensional array. Initially, only one site at the middle of the array is occupied by a single agent. The rates for either diffusion, birth and the "per pair" coalescence are predetermined and kept constant along the simulation. At each time step any single particle was surveyed, the number of other particles that may perform "pair coalescence" with this individual is enumerated, and its chance to disappear was calculated. With that, the relative chances for the next event (yield of an offspring, migration or coalescence) is calculated, and the event to happen is determined by "tossing" a weighted coin. If the event selected is migration or birth, one of the particles is selected at random and perform this operation. In case of coalescence, the probability of a single particle to disappear is proportional to the number of particles in its  $L$  neighborhood.

Iterating this process and recording the location of the rightmost particle in the system, one gets the instantaneous location of the front. This noisy quantity has to be average over many realization of the other processes with the same parameters, and the results, shown in Figure 1, clearly indicate that, unlike its continuum approximation, the discrete process allows for the nonlinear competition term to effect the propagation velocity, so the front movement rate is diminishing as the interaction growth. However, even for large competition the front location still grows linearly with time, so the invasion is still ballistic, not diffusive or even superdiffusive.

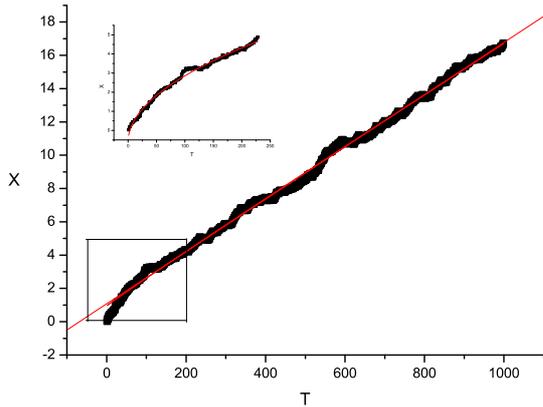


FIG. 2: The average (over 200 samples) location of the front for top hat competition. At short time the propagation is sublinear  $x \sim t^\alpha$  while for long time the propagation is linear  $x \sim t$ . The simulation parameters are:  $D = 0.5$ ,  $\alpha = 0.5$  and  $\gamma = 2$ . The inset emphasizes the short time behavior.

The underlying mechanism beyond that result is related to the combination of diffusion and "escape" from the competition range. For any growth and competition parameters there is a finite probability that a new born reactant will diffuse far enough to escape the competition length and give birth to another new reactant there. The event of birth and diffusing out of the interaction length has a typical time  $\tau$ . This new time scale determines the effective growth rate, as now a reactant gives birth at a rate  $a + 1/\tau$  (rather than  $a$  in the case with local competition). Therefore the overall propagation mode turns to be of a Fisher type, i.e., ballistic. Averaging over many samples one indeed find a sublinear displacement of the front,  $x \sim t^\alpha$  with  $\alpha < 1$ , at short times, but after some transient period the propagation becomes linear in time  $x \sim t$  as shown at figure (2)

The dependence of the linear velocity on the competition strength for top hat interaction is shown in figure (3). The functional dependence of  $v$  on  $\gamma$  is not trivial, and is clearly neither exponential nor a power law (see inset). We have also failed to fit the behavior to stretched exponent. It seems that the velocity is determined by few processes that interfere with each other. Among these one can point out the reduction of the average density of particles, yielding larger corrections to the front velocity due to discretization [8, 10], and the suppression of the growth rate for the leading particle by its neighbors along the front. It is interesting to note that the "break" between the two regions in the inset of Figure 3 happens when the average number individuals at a site is of order unity. (The number of individuals at site is  $a/(L\gamma)$ ; for the parameters of (3) it is about one when  $\gamma = 0.1$

The effective growth rate depends, as explained before, on the competition strength (this is the chance for an offspring to diffuse out of the competition length before

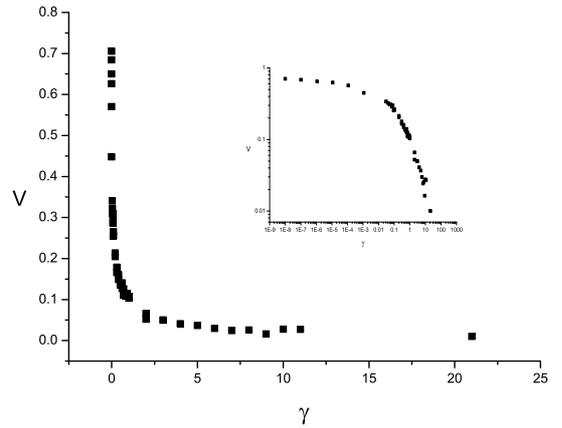


FIG. 3: Front velocity (arbitrary units) of the discrete Fisher process with long range competition versus the interaction strength  $\gamma$ . The kernel type is top hat ( $L = 5$ ), the diffusion coefficient is  $D = 0.25$  and growth rate is  $\alpha = 0.5$ . In the inset the same data is presented in log-log scale, indicating that the decay is faster than a simple power law. However, the decay is slower than exponential and does not fit to simple stretched exponent. At the large concentration limit (small  $\gamma$ ) the velocity converges to its "mean field" value with the appropriate corrections coming from the spatial discretization as predicted by [22].

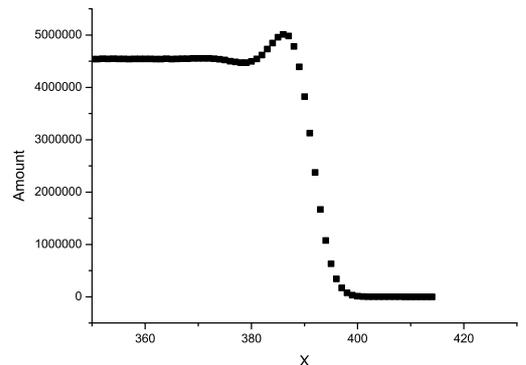


FIG. 4: The shape of a Fisher front close to the mean field limit, i.e., where the number of individual reactants at saturation is large. The main features of the front, including its leading tail and the "overshoot" bump, correspond to the equivalent characteristics for the nonlocal FKPP model as shown in [13]. The simulation parameters are  $a = 0.5$   $D = 0.25$   $L = 5$   $\gamma = 10^{-8}$

it undergoes coalescence), thus the nonlocal competition rescales the growth rate. Note that at the small  $\gamma$  limit the front velocity converges to its mean field value and the front shape is similar to the shape observed in the continuum case (see Fig. 4).

To conclude, The branching-coalescence process with

nonlocal coalescence of individual agents is mapped, in the mean field approximation, to the FKPP equation with nonlocal competition. Our numerical analysis shows that, while the nonlocal competition may effect the invasion of the stable phase into the unstable region, it can

not change the fundamental characteristic of the front propagation, namely, its ballistic motion. Based on the general argument of renormalization of the growth rate by the escape time, we expect this feature to be valid as long as the nonlocal competition length is finite.

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