Multiplicative Noise and Second Order Phase Transitions

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The scale-free distribution of cluster sizes in continuous phase transitions is linked to the law of proportional effect. A numerical study of a two-dimensional Ising model suggests that a cluster size undergoes a multiplicative birth-death process. At the transition the ratio between birth and death rates approaches unity for large clusters, and the resulting steady state shows a power-law behavior. The percolation dynamic, on the other hand, yields a geometric phase transition without ergodicity breaking, where large-scale merging and splitting of clusters dominate the distribution. Instead of short-range birth-death jumps, the percolation transition is characterized by Lévi flights along the cluster-size axis.

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Power laws, or otherwise fat-tailed statistics, are ubiquitous in both natural and human-made complex systems [1]. This phenomenon is somewhat surprising: given complex interactions between many objects one can naively expect the competing microscopic mechanisms to produce a Gaussian statistic, in agreement with the central limit theorem. The appearance of fat-tailed distribution is thus nontrivial, and the attempts to clarify and to classify the mechanisms beyond this behavior comprise a substantial part of the modern physics literature.

One of the basic suppositions in that field is the distinction between equilibrium and out-of-equilibrium systems. In equilibrium systems one has to fine-tune the external parameters (e.g., temperature or pressure) to reach criticality, i.e., the point of second order transition, where a power-law distribution is observed. If the system is not at equilibrium, on the other hand, a few underlying mechanisms may yield scale-free statistics. In many cases this behavior persists for a wide range of parameters, giving rise to the concepts of "self-organized" or "robust" criticality [1–3].

Among many scenarios that lead to robust critical behavior, the most generic is the one involving an unbiased random walk where the jump rate, or the step size, scales linearly with the distance from the origin [4]. The simplest example is the return of stock-market investments. To a first approximation (and sometimes even beyond that), a stock market is a generator of random interest rates; thus, the value of a stake follows a random walk, where the step size is proportional to the value. This corresponds to a standard diffusion in log space, and thus the return statistic is log-normal. With slight modifications [5–8], the multiplicative noise yields a power law. This behavior was observed long ago in the context of econo- and ecosystems, and is also known as the law of proportional effect [4].

The aim of this Letter is to reveal the connection between this type of multiplicative noise and the power law that characterizes the correlation length at the point of second order transition. Following recent works on patch dynamics and robust criticality in ecosystems [9,10], we turn here to consider the same features in two archetypical models of second order transition, an Ising model with a nonconserved order parameter and site percolation, both in two spatial dimensions on a square lattice. For percolation we have implemented a simple dynamics where a site, chosen at random, becomes occupied with probability p and empty with probability $1 - p$, independent of its prior state and its neighbors states. In both cases we have monitored the dynamics of *clusters*, i.e., connected groups of spin-up (Ising) or occupied sites (percolation). A cluster is defined as a group of sites that are spatially configured in a way that enables one to reach every member of the group from every other member by means of nearest neighbor jumps between sites that are included in the cluster.

Both Ising dynamic (either the realistic or its formulations used in numerical analysis, like Metropolis algorithm) and the percolation process are indeed Markov chains, with well defined rules dictating the chance for transition from one microscopic state to another. In order to track the cluster dynamics we first project this Markov process on the cluster-size space, as illustrated in Fig. 1. A new Markov matrix Q is defined, where $Q_{n,m}$ is the rate in which clusters of size n become clusters of size m. $Q_{n,m}$ has been obtained numerically by monitoring the time evolution of clusters at equilibrium. At thermodynamic equilibrium this chain should admit a single, aperiodic, recurrent class; thus, the patch distribution must converge to its stationary distribution, π_n, which is (up to normalization) the invariant eigenvector of Q.

What is the structure of the matrix Q? In Fig. 1 this matrix is shown for the Ising dynamics at $T = 14.8$ (all simulations preformed at $J = 1$ and with no external magnetic field; the transition occurs at $T_c = 2.269$). It turns out that the Markov chain is very close to a birth-death process: given a size n cluster, the rate of transitions to $n + 1$ and $n - 1$ is much higher than all other transition rates. The reason for that lies in the topology of the Ising clusters. These clusters are relatively packed, with fractal dimension $D = 91/48$ in $2d$ [11]. In an analogy with the concept of

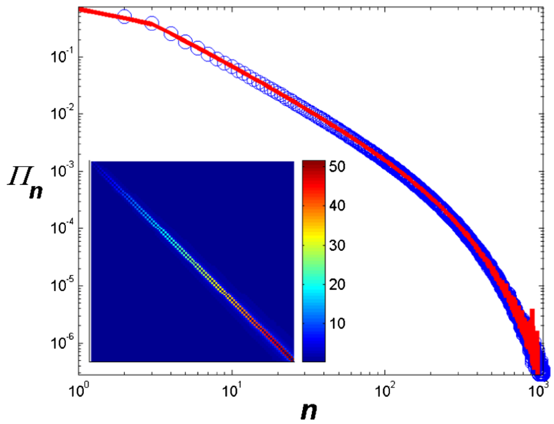

FIG. 1 (color online). Projection of the Ising dynamics on the cluster space. The transition matrix Q (inset, color coded the values of $Q_{n,m}$ for n, $m = 1$–100, arb. units) has been constructed by monitoring the evolution of clusters mass along time (the simulation used Metropolis algorithm in $2d$, $T = 14.8$). The invariant eigenvector of Q reflects the steady state statistics of cluster sites: π_n, the (normalized) nth element of this eigenvector (red line) indeed fits the actual patch distribution measured directly from the simulation (blue circles).

"red" sites [11,12], we define here "yellow," or connecting sites, as sites in which their removal decomposes a single cluster (not necessarily the infinite cluster) into two (or more) disconnected patches. The abundance of yellow sites reflects the relative importance of merging (splitting) (multicluster) processes, as opposed to a continuous growth (shrinkage) of an individual patch. Our simulations (Fig. 2) indicate that in the off transition regime the number of yellow sites in a cluster, m_y, scale linearly with the mass of a large cluster, $m_y/n \to \zeta(T)$. In the transition regime $\zeta(T)$ approaches zero like $(T - T_c)^{0.56}$; the dependence of

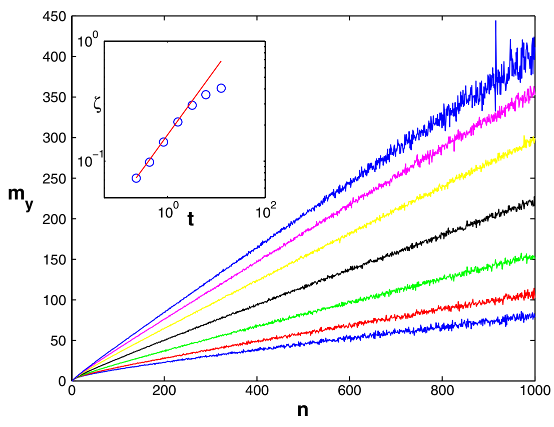

FIG. 2 (color online). Number of yellow bonds, m_y, vs the numbers of sites in a cluster for different temperatures ($T = 14.8, 8.4, 5.5, 3.9, 3.1, 2.7, 2.5$ from the top line downwards). The slope of the linear part decreases as the reduced temperature, $t \equiv (T - T_c)/T_c$, approaches zero. The inset shows the slope ζ vs t on double logarithmic scale, together with the line $\zeta \sim t^{0.56}$.

m_y on the mass of the cluster at the transition is thus sublinear, and the approximation of Q by a triangular matrix becomes better and better. Adopting the birth-death processes convention we define $d_n \equiv Q_{n,n-1}$ and $b_n \equiv Q_{n,n+1}$.

We should stress that this omission of all high-order merge-split processes is only an approximation. Close to the transition the density of yellow sites do not scale linearly with the mass of the cluster, but it still may grow sublinearly. Moreover, the small scale processes are clearly nonuniversal, and perhaps changes in the microscopic dynamic may yield a more general band diagonal Q matrix. The results suggest, though, that the random walk along the cluster mass axis is characterized by short-range jumps, so under appropriate coarse graining the process is a birth-death one. This is in sharp contrast with the *rate* of the process that grows with the mass of the droplet. As emphasized below, the birth-death approximation works very well even far from the transition.

Neglecting the yellow sites one may facilitate the retrieval of the transition rates: by looking at uncorrelated snapshots of the Ising system and identifying the clusters it is easy to evaluate b_n and d_n directly for any cluster from the transition probability of any spin in the cluster or on its external perimeter. The results, shown in Fig. 3, resemble very much the situation described in [10] for the out-of-equilibrium, robust criticality case: both b_n and d_n are linearly increasing functions of n, with a constant bias towards extinction, i.e., $b_n = \gamma n$, $d_n = b_n + \alpha n + \Delta$, where α_n vanishes at the transition. The relative abundance of mass-n clusters satisfies, at the steady state, the local

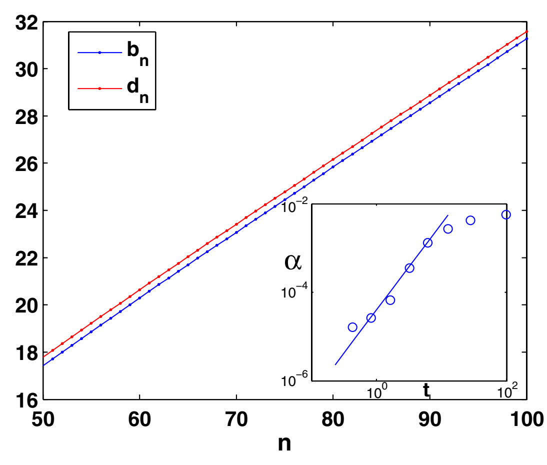

FIG. 3 (color online). Close to the transition (here $T = 3.1$), the Ising cluster dynamics follows a birth-death process. This is a multiplicative random walk biased towards extinction: for any n, d_n is larger than b_n by $\Delta_n = \Delta + \alpha n$. As seen in the inset, α approaches zero at the transition, yielding a scale-free patch-size distribution. Around criticality (2) implies $\alpha \sim n_0^{-1}$; on the other hand $n_0 \sim \xi^{d_f} \sim t^{-\nu d_f}$. One thus expects $\alpha \sim t^{\nu d_f}$ as shown by the full line plotted for the known values of d_f and ν for $2d$ Ising model (1.95 and 1, respectively).

balance equation, $\pi_n b_n = \pi_{n-1} d_{n-1}$. This implies that π_n can be obtained analytically [10]:

$$\pi_n = \frac{b_0 \pi_0}{d_n} \prod_{m=1}^{n-1} \frac{b_m}{d_m} \propto \frac{\Gamma(n)\Gamma(1+\frac{\Delta}{\gamma+\alpha})}{\Gamma(1+n+\frac{\Delta}{\gamma+\alpha})}\left(\frac{\gamma}{\alpha+\gamma}\right)^n. \quad (1)$$

The product (1) yields, in the limit $n \gg 1$, a truncated power law,

$$\pi_n = A n^{-(1+\Delta/(\alpha+\gamma))} e^{-n/n_0}, \quad (2)$$

where A is a normalization constant and $n_0 \equiv [\ln(\frac{\alpha+\gamma}{\gamma})]^{-1}$ is the cluster size above which π_n decays exponentially. The value of α dictates the distance from the transition and the typical mass above which clusters are rare. Unfortunately, the measurement of α is difficult in the vicinity to the transition: beyond the problem of critical slowing down, the values of b and d are very close. Yet, in the inset of Fig. 3 the decay of α towards zero is demonstrated, showing a reasonable agreement with the known fractal dimension of the $2d$ Ising clusters.

Although the approximation of treating the cluster oriented dynamics as a pure birth-death process is formally justified only in the transition regime, our numerics shows that Eq. (2) gives a good estimation of the cluster-size distribution even for temperatures substantially larger than T_c (Fig. 4).

The divergence of the typical size of an Ising clusters turns out, thus, to be a manifestation of the law of proportional effect. Indeed, any birth-death process yields a heavy tail distribution if $d_n/b_n \to 1$ as $n \to \infty$. Moreover, if $d_n = b_n + \Delta$ and b_n scales like n^β the value of β dictates the asymptotic of π_n. For $\beta > 1$ π_n falls like a power law and the power is independent of Δ, while $\beta < 1$ yields a stretched exponential behavior. Here, as the density of connecting sites approaches zero close to the transition, the cluster dynamics obeys a birth-death process with $\beta = 1$; in this marginal case π_n decays like a power law that depends on Δ; a similar behavior has been observed by [13] for the zero range process.

The situation is different for the case of percolation dynamics. Here (Fig. 5) merge-split processes are important even at the transition point. Approximating the Q matrix by its tridiagonal part one observes no special feature at the transition; instead, the random walk in the patch-size space becomes a *Lévy flight*, for which the chance of a size s jump falls like a power law with s; For the percolation transition, the density of the red bonds approaches zero very slowly (inversely proportional to the correlation length [14]); thus, the spanning cluster still contains red sites that may flip in time, inducing huge jumps along the mass axis.

We have demonstrated, thus, two routes to criticality for two specific models. Of course, the field of equilibrium continuous phase transitions is much wider, and in order to rigorously suggest a general classification of cluster dynamics one should survey other systems, like the general q states Potts model, magnetic systems with conserved order parameter, system at different dimensionality, and much more. Still at this stage of the research we want, in the next two paragraphs, to present a conjecture supported by a plausibility argument about the general picture.

The percolation transition is geometric; there is no "attraction" that keeps the cluster sites together. This property manifests itself in the sparse topology of the clusters and particularly in the presence of a substantial density of yellow bonds close to the transition. As a result, at the transition point the cluster dynamics becomes a Lévi flight. The Ising model, on the other hand, is an example of a "real" thermodynamic system in which the clustering

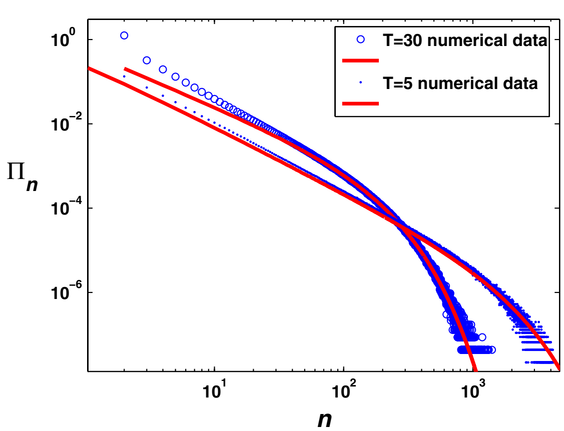

FIG. 4 (color online). Cluster-size distributions for two different temperatures are presented. The dots and circles correspond to distributions that have been measured directly from Ising systems at equilibrium. The thick lines stand for the corresponding distributions obtained from Eq. (2).

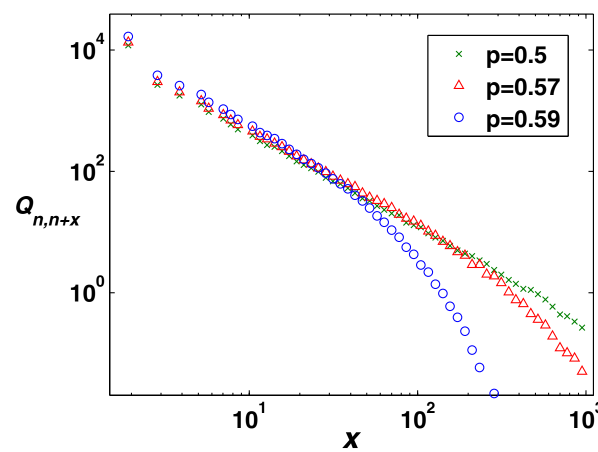

FIG. 5 (color online). $Q_{n,n+x}$ is plotted vs x for site percolation dynamics on a square lattice. For $p < p_c$ the chance of a size x jump (a patch of size n becomes $n + x$) follows a truncated power law; under appropriate rescaling the system behaves as a birth-death process. As the system approaches $p_c \cong 0.593$, on the other hand, there is a scale-free distribution for these merge-split events.

reflects the attractive force that keeps cluster sites together. In such a case, large-scale merge-split processes become negligible close to the transition point and the power-law droplet statistics are established by the law of proportional effect.

The topological differences between Ising and percolation clusters seem to reflect the difference between thermodynamic and geometric transitions. The phase transition in a magnetic system is a manifestation of ergodicity breaking: if an infinite system is initiated with all spins up, its evolution below T_c is restricted to the domain where the magnetization is positive. Down spin clusters cannot grow indefinitely since their perimeter admits overall positive curvature and the effective field close to the surface points upwards. Above T_c the same physics holds and an isolated island still tends to shrink, the system acquires its ergodicity only due to the existence of merge (split) processes. Thus in a thermodynamic transition one expects that the density of yellow bonds, at least above some typical cluster size, will be negligible at and below the transition; there are only finite size jumps on the cluster-size axis, and the power law is related to the multiplicative character of the cluster's dynamic.

The structure of clusters and droplets in systems that undergo a continuous phase transition has attracted a lot of attention along the years. Insights that appeared in the context of scaling and renormalization group theory were implemented to show equivalence between different models and to facilitate numerical simulation [15–18]. Two-dimensional models, in particular, are known to be conformally invariant at the transition point in the continuum space limit. This property allows one to prove some exact results and it lies in the foundation of the modern theory of cluster topology, based on the stochastic Loewner evolution (for a review, see [19]). The results presented here suggest a connection between the topology of the droplet, its local dynamics, and the global geometry close to, and far away from, criticality. Recently, confocal microscopy techniques appeared allowing real-time tracing of the clusters formed in a colloidal suspension up to the single particle level [20]. This opens the experimental possibility to extract the Q matrix terms from the observed cluster dynamic and to fit the overall droplet size distribution with the parameters of the droplet's multiplicative random walk dynamics.

To conclude, our numeric provides a phenomenological connection between multiplicative random walks and the emergence of criticality in thermodynamic second order transitions. Clearly, this observation does not *explain* the appearance of scale-free distributions and cannot provide an alternative to the well-known analytic tools like renormalization group technique. Still, the integration between the two branches of complexity science opens a new conceptual framework that may be used to predict and to classify complex patterns in general.

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