On the group theory of the polarization states of a massless field

N. Shnerb
Department of Physics, Bar-Ilan University, Ramat-Gan, Israel

L. P. Horwitz[a)]
School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540

(Received 12 July 1993; accepted for publication 9 December 1993)

It is shown that the theory of representations of the Poincaré group applied to the vector potentials of a massless field yields in a simple and direct way (without assuming that the field is a gauge field, and without detailed assumption on the form of the equation of motion), the structure of the polarizations, the Gupta–Bleuler condition, and gauge invariance of the theory. The method is shown to apply to the Maxwell field and to a five-dimensional generalization of the Maxwell field (which properly contains the Maxwell theory) associated with manifestly covariant dynamics.

I. INTRODUCTION

In this paper we study the group theoretical structure of the Maxwell radiation field as a guide to the construction of the polarization states of a higher-dimensional generalization of the Maxwell field. Such a generalization is required to describe the electromagnetic interactions of the manifestly covariant mechanics of Stueckelberg[1] and Horwitz and Piron.[2] This manifestly covariant relativistic classical and quantum mechanics has for its fundamental dynamical constituents, as for Feynman's formulation of quantum electrodynamics,[3] objects defined locally in space-time. The states of such a system evolve with an invariant parameter τ, which we take to be a universal time, as that of Newton.

Since the conserved current of the Maxwell theory is an integral over the world line of local contributions,[4,5] e.g., in the classical case,

$$J^\mu(x)=\int_{-\infty}^{\infty} e\frac{p^\mu(s)}{m}\,\delta^4(x-x(s))ds, \tag{1.1}$$

where s is the proper time of the charged source and $J(x)$ is the usual four-current, and we write x for $x^\mu=(x,t)$. To construct this current, the world-line must be known. The problem of describing the motion of a system of more than one particle, with interactions mediated by the Maxwell electromagnetic field, is therefore not well posed. An iterative procedure to find a self-consistent solution may become unstable. One therefore seeks to formulate a more general theory of electromagnetic interaction which includes the Maxwell theory to account for the known phenomena associated with Maxwell radiation, but also provides the basis for a well-posed system of equations. Such a theory is provided by the requirement that the Stueckelberg–Schrödinger equation be of gauge-invariant form.

Let us consider the case of one particle in interaction with the electromagnetic field. If the wave function $\psi_\tau(x)$ undergoes a local phase transformation, $\psi_\tau(x)\rightarrow e^{ie'\Lambda(x,\tau)}\psi_\tau(x)$, then in the free motion equation ($\partial_\tau\equiv\partial/\partial\tau$, $\partial_\mu\equiv\partial/\partial x^\mu$)

[a)]Permanent address: School of Physics, Tel-Aviv University, Ramat-Aviv, Israel; also at Department of Physics, Bar-Ilan University, Ramat-Gan, Israel.

0022-2488/94/35(4)/1658/10/$6.00

$$i\partial_\tau \psi_\tau(x) = -\frac{\partial_\mu \partial^\mu}{2m} \psi_\tau(x) \tag{1.2}$$

along with the replacement of $-i\partial_\mu$ by $-i\partial_\mu - e'a_\mu(x,\tau)$, one must replace $i\partial_\tau$ by $i\partial_\tau + e'a_5(x,\tau)$. [The a_5 field plays the role of an effective local mass shift for the matter fields, reflecting the fact that the photon is off mass shell due to the interaction. Its Fourier spectrum is the off mass shell mass of the photon. It is required by the gauge invariance of the Stueckelberg equation. See also Shnerb and Horwitz[7] for further discussion.] The corresponding gauge transformation[6]

$$a_\alpha \rightarrow a_\alpha + \partial_\alpha \Lambda, \tag{1.3}$$

for $\alpha = 0,1,2,3,5$ preserves the form of the equation

$$(i\partial_\tau + e'a_5)\psi_\tau(x) = \frac{1}{2M}(\partial_\mu - ie'a_\mu)(\partial^\mu - ie'a^\mu)\psi_\tau(x). \tag{1.4}$$

From this equation, one sees that the current $j_\tau^\alpha(x) = (j^\mu{}_\tau(x), \rho_\tau(x))$, where

$$j_\tau^\mu(x) = -\frac{ie'}{2M}\{\psi_\tau^*(\partial^\mu - ie'a^\mu)\psi_\tau - (\partial^\mu + ie'a^\mu)\psi_\tau^*\psi_\tau\},$$

$$\rho_\tau(x) = |\psi_\tau(x)|^2, \tag{1.5}$$

satisfies the (five-dimensional) continuity equation[1]

$$\partial_\alpha j^\alpha = \partial_\mu j_\tau^\mu + \partial_\tau \rho_\tau = 0. \tag{1.6}$$

Equation (1.4) can be generalized to two or more particles interacting through the fields $a_\tau^\alpha(x)$ by summing over the terms $(1/2M_j(\partial_{\mu j} - ie'a_\mu(x_j,\tau))(\partial_j^\mu - ie'a^\mu(x_j,\tau))$, and on the left-hand side, introducing the sum $e'(a_5(x_1,\tau) + a_5(x_2\tau))$. It is then clear that the problem of the evolution of the system is, in principle, well posed.

The equations of motion for the radiation field, derived from a Lagrangian which also implies the Stueckelberg–Schrödinger equation and includes the term $\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta}$ for the kinetic terms of the field equations

$$\partial_\alpha f^{\beta\alpha} = j_\tau^\beta(x), \tag{1.7}$$

where $(\partial_5 \equiv \partial_\tau)$,

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha. \tag{1.8}$$

The conservation law (6) is consistent with the antisymmetric form of $f^{\beta\alpha}$. In the absence of matter, the symmetry of the homogeneous equation is $O(3,2)$ or $O(4,1)$ (as we shall review later) depending on the signature one chooses for the 5th component of the metric tensor. The stable modes of the classical field for spacelike four-momenta correspond to the $O(3,2)$ metric and for the timelike components, the $O(4,1)$ metric; from the point of view of the corresponding Maxwell theory, the five-dimensional fields are clearly not on the four-dimensional zero mass shell.[6] In order to carry out a quantization of the off shell fields, it is essential to represent the fields in terms of the physical polarization components.

The construction of the polarization components has, as we shall see, a group theoretical interpretation which can be used as a general framework. We first consider the Maxwell theory and then generalize the result to the five-dimensional case (the general N dimensional case will be discussed elsewhere). Weinberg[8] has shown that the stabilizer group of the massless four-momentum vector plays an essential role in deriving current conservation, and even the Maxwell equations, from the S-matrix. Shaw[9] has discussed the structure of the representations of the massless vector field, and showed, in particular, that the $\pm$ helicity states lie in the subspace of functions satisfying the Lorentz gauge condition modulo the functions that are pure gauge. Our discussion goes in the opposite direction from Shaw;[9] we show constructively that the vector field must be a gauge field. In this paper we derive both the Gupta–Bleuler conditions and the necessity for the gauge freedom of the theory.

II. POLARIZATION SYMMETRY OF THE MAXWELL THEORY

Consider the Fourier representation of the free electromagnetic field $A^\mu(k)$, where, since the field is massless, $k^\mu k_\mu=0$. The Lorentz transformation properties of the field are given by

$$A'^\mu(k)=\Lambda^\mu_\nu A^\nu(\Lambda^{-1}k). \tag{2.1}$$

In particular, for the subgroup of the Lorentz group which leaves k^μ invariant,

$$A'^\mu(k)=\Lambda^\mu_\nu A^\nu(k), \tag{2.2}$$

so that on this stabilizing vector, A^μ is a representation of this subgroup. The subgroup of $O(3,1)$ which leaves a lightlike vector invariant is $E(2)$, and to find the form of the vector field $A^\mu(k)$, we must study the representations of $E(2)$. Since $E(2)$ is solvable, all of its finite dimensional irreducible representations are one dimensional.[10]

The generators of this $E(2)$ are two operators equivalent to generators of translation in their algebraic properties, and the generator of a rotation around the space part of k^μ. For ($\mu=0,1,2,3$, metric $-,+,+,+$) let us take

$$k^\mu=(k,0,0,k). \tag{2.3}$$

Then the generators which stabilize this vector are

$$J_3,\quad L_1=K_1+J_2,\quad L_2=K_2-J_1, \tag{2.4}$$

where (in the form of the first index contravariant and the second covariant)

$$J_3=i\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix},\quad L_1=i\begin{pmatrix}0&1&0&0\\1&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad L_2=i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&-1\\0&0&1&0\end{pmatrix}. \tag{2.5}$$

The algebra of $E(2)$,

$$[L_1,L_2]=0,\quad [J_3,L_1]=iL_2,\quad [J_3,L_2]=-iL_1 \tag{2.6}$$

can be represented in one dimension only if L_1 and L_2 are represented by zero, i.e., the translation part of $E(2)$ must be trivial. [In his proof of charge conservation and the equivalence of gravitational and inertial mass, Weinberg[8] also used the necessity of trivial representation of the translations in finite-dimensional representations of $E(2)$.[11]] Since the matrices L_1 and L_2 are nilpotent, we can write the group transformations for the translations as

$$T_1(\alpha)=e^{-i\alpha L_1}=\begin{pmatrix} 1+\frac{\alpha^2}{2} & \alpha & 0 & \frac{-\alpha^2}{2} \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{\alpha^2}{2} & \alpha & 0 & 1-\frac{\alpha^2}{2} \end{pmatrix},$$

$$T_2(\beta)=e^{-i\beta L_2}=\begin{pmatrix} 1+\frac{\beta^2}{2} & 0 & \beta & -\frac{\beta^2}{2} \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{\beta^2}{2} & 0 & \beta & 1-\frac{\beta^2}{2} \end{pmatrix}. \tag{2.7}$$

Then

$$T_1(\alpha)^{\nu}_{\mu}A^{\mu}=\begin{pmatrix} A_t+\frac{\alpha^2}{2}\Delta+\alpha A_x \\ A_x+\alpha\Delta \\ A_y \\ A_z+\frac{\alpha^2}{2}\Delta+\alpha A_x \end{pmatrix}, \quad T_2(\beta)^{\nu}_{\mu}A^{\mu}=\begin{pmatrix} A_t+\frac{\beta^2}{2}\Delta+\beta A_y \\ A_x \\ A_y+\beta\Delta \\ A_z+\frac{\beta^2}{2}\Delta+\beta A_y \end{pmatrix}, \tag{2.8}$$

where $\Delta=A_t-A_z$.

We require that the vector A^{μ} must be stable under the action of $T_1(\alpha),T_2(\beta)$ for arbitrary α,β. If we require that the elements of the vectors (2.8) are equal to those of the original one, then Δ, A_x, and A_y must vanish, and we obtain a trivial solution for which $F^{\mu\nu}(k)=i(k^{\mu}A^{\nu}-k^{\nu}A^{\mu})$ is zero. If $\Delta\neq 0$, then one must identify $A_x+\alpha\Delta, A_y+\beta\Delta$ as in an equivalence class with A_x and A_y. [Barut and Raczka[12] discuss the structure of the representations of the Poincaré group for the massless case from the point of view of induced representations, as we do here. Using the gauge freedom of the field, they impose the Lorentz condition [for the vector field $P^{\mu}\phi_{\mu}(P)=0$ and the gauge condition $\phi_{\mu}(P)\rightarrow\phi_{\mu}(P)+P_{\mu}\lambda(P)$]. They remark that this set of functions forms an equivalence class. If H_1 is the set of functions in H which have measurable (on the Poincaré group) scalar product with the (in this case rank one) tensors which carry finite dimensional representations of the stability group, and which transform themselves with a finite dimensional faithful indecomposable representation of a covering of $E(2)$ then one may realize a unitary representation of the Poincaré group in the quotient space H/H_1. Generalizations are discussed for higher spin fields. In our construction, we do not impose the gauge symmetry or the Lorentz condition, but derive these from the necessary structure of the equivalence classes. Our assumptions are equivalent to those of Weinberg[8] who follows a somewhat different method, involving direct computation, than we do.] Furthermore, the t and z components must be identical with equivalence classes including combinations of Δ, A_y, and A_x. The field is thus trivially equivalent to zero, i.e., the necessary equivalence class is too large. Alternatively, if $\Delta=0$, then the required equivalence classes are

$$A_t\sim A_t+\beta A_y+\alpha A_x, \quad A_z\sim A_z+\beta A_y+\alpha A_x.$$

This freedom corresponds to the gauge transformation associated with

$$A^\mu \sim A^\mu + \partial^\mu \Lambda,$$

which in Fourier transform is

$$A^\mu(k) \sim A^\mu(k) + ik^\mu \Lambda(k)$$

for the vector (2.3).

Hence the choice $\Delta=0$ provides a nontrivial realization of the vector potential. This condition, that

$$\Delta = A_t - A_z = 0$$

is precisely the Gupta–Bleuler gauge condition on the physical states, i.e., that any physical state must be constructed from linear combinations of $(a^\dagger{}_t + a^\dagger{}_z)^n$ for any integer $n \geqslant 0$ on the photon vacuum.[13] We have shown that this requirement follows entirely from the group theoretical properties of a massless ($k^\mu k_\mu = 0$) vector field. Furthermore, we have shown that the transformation properties of a nontrivial vector field of this type is defined up to an equivalence class corresponding to the gauge freedom of the theory. The physical polarization states are the components A_x, A_y orthogonal to the **k** vector, which cannot be altered by local gauge transformation. They serve as a representation for SO(2), the little group by which a representation of $O(3)$ can be induced along an orbit labeled by **k**. This $O(3)$ corresponds to the transformations of the Poynting vector, the transport of energy in the Galilean frame in which the energy and momentum of the Maxwell fields are identified.

III. POLARIZATION STATES FOR THE FIVE-DIMENSIONAL FIELD

For the free five-dimensional field, the five-momentum, containing k^μ and κ, the "momentum" conjugate to τ, is on the zero mass shell. It follows from Ref. 6 that in the absence of sources,

$$(\partial_\tau \partial^\tau + \partial_\nu \partial^\nu) f^{\alpha\beta} = 0. \tag{3.1}$$

As remarked in Ref. 6, the Fourier transform of (3.1) in space-time variables results in

$$(\partial_\tau \partial^\tau - k_\nu k^\nu) f^{\alpha\beta}(k,\tau) = 0. \tag{3.2}$$

For the stability of the free field solutions, we see from (3.2) that for $k_\nu k^\nu > 0$ (spacelike) $\partial_\tau \partial^\tau$ must be $-\partial^2{}_\tau$, and for $k_\nu k^\nu < 0$ (timelike), $\partial_\tau \partial^\tau$ must be $+\partial^2{}_\tau$. These correspond to the symmetries $O(3,2)$ (spacelike k^μ) or $O(4,1)$ (timelike k^μ) for the free radiation field.

Taking the Fourier transform of (3.2) over τ, with conjugate variable κ, one finds

$$(\pm\kappa^2 - k_\nu k^\nu) f^{\alpha\beta}(k,\kappa) = 0, \tag{3.3}$$

where the $\pm$ corresponds, respectively, to $O(3,2)$ and $O(4,1)$. It therefore follows that the five-dimensional fields are "massless" on a five-dimensional momentum space.

Let us first treat the case of $O(4,1)$, where we assume

$$k_\alpha k^\alpha = -\kappa^2 - k^2 + (k^0)^2 = 0. \tag{3.4}$$

The symmetry subgroup of $O(4,1)$ which leaves the vector (k^0,k^1,k^2,k^3,κ) invariant is $E(3)$. We see this by taking, for example,

$$k^\alpha = (k,0,0,k,0). \tag{3.5}$$

The stabilizing subgroup of this k^α is larger than SO(3), since $k^0=k^3$. In fact, we have the additional symmetries generated by

$$L_1=K_x-J_{xz}, \quad L_2=K_y+J_{yz}, \quad L_3=K_z-J_{\tau z}, \tag{3.6}$$

where (the boost generators are designated with one index; the second, that of t, is understood)

$$K_x=i\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad K_y=i\begin{pmatrix}0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$K_z=i\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad J_{xz}=i\begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\end{pmatrix}, \tag{3.7}$$

$$J_{yz}=i\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}, \quad J_{\tau z}=i\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-1\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix},$$

are the boost and rotation generators of $O(4,1)$ that enter in the little group. From the commutation relations

$$[J_{xz},J_{yz}]=-iJ_{xy}, \quad [K_x,K_y]=-iJ_{xy},$$

and

$$[K_x,J_{yz}]=[K_y,J_{xz}]=0,$$

it follows that

$$[L_1,L_2]=0. \tag{3.8}$$

Similarly,

$$[L_1,L_3]=[L_2,L_3]=0. \tag{3.9}$$

The group generated by L_1, L_2, L_3, J_{xy}, $J_{x\tau}$, $J_{y\tau}$ is $E(3)$. This group is a semidirect product of the translation part, which is a maximal simply connected solvable normal subgroup and a simply connected semisimple subgroup, and is not solvable; however, one can show that the "translation" part of this group must be represented trivially in any finite dimensional representation of such a group (Ref. 10, p. 227). As in the $O(3,1)$ example of the Maxwell field, the matrices representing L_1, L_2, L_3 are nilpotent, e.g.,

$$L_1=\begin{pmatrix}0&1&0&0&0\\1&0&0&-1&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix},\quad L_1^2=\begin{pmatrix}1&0&0&-1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&-1&0\\0&0&0&0&0\end{pmatrix},\quad L_1^3=0. \tag{3.10}$$

It then follows that

$$T_1(\alpha)=\begin{bmatrix}1+\frac{\alpha^2}{2}&\alpha&0&-\frac{\alpha^2}{2}&0\\ \alpha&1&0&-\alpha&0\\0&0&1&0&0\\ \frac{\alpha^2}{2}&\alpha&0&1-\frac{\alpha^2}{2}&0\\0&0&0&0&1\end{bmatrix}. \tag{3.11}$$

and similarly for $T_2(\beta),T_3(\gamma)$.

We now require that the action of the translations on the five-vector a^α be trivial, i.e., that

$$T_1(\alpha)a_\alpha=\begin{bmatrix}a_0+(\alpha^2/2)\delta+\alpha a_1\\a_1+\alpha\delta\\a_2\\a_3+(\alpha^2/2)\delta+\alpha a_1\\a_5\end{bmatrix}, \tag{3.12}$$

where $\delta=a_0-a_3$.

Similar structures are induced by $T_2(\beta),T_3(\gamma)$. By the same argument we gave in the Maxwell case, we see that we must have $\delta=0$, i.e.,

$$a_0-a_3=0, \tag{3.13}$$

and conclude as well that we have the equivalence classes

$$a_0\sim a_0+\alpha a_1+\beta a_2+\gamma a_5,\quad a_3\sim a_3+\alpha a_1+\beta a_2+\gamma a_5. \tag{3.14}$$

Clearly the structure of the stable vector (3.5) admits this equivalent class in the gauge group,

$$a^\alpha\sim a^\alpha+ik^\alpha\Lambda(k). \tag{3.15}$$

The components a_0, a_3 are analogous to the timelike and longitudinal modes of the Maxwell field, and a_1, a_2, and a_s are analogous to the transverse modes and span a representation for the polarization symmetry group $O(3)$. This $O(3)$ induces a representation for $O(4)$ under the action of $O(4)/O(3)$.

For the $O(3,2)$ case, the little group is $E(2,1)$. We must specify for $O(3,2)$ two "longitudinal" directions, one in the timelike (two-dimensional) subspace, and one in the spacelike (three-dimensional) subspace. Let us choose for these, without loss of generality, the 0 and 3 directions. We shall stabilize the vector $(k,0,0,k,0)$ as before, but now

$$L_1=K_{xt}+J_{zx},\quad L_2=K_{ty}+J_{yz},\quad L_3=K_{z\tau}-J_{zt}, \tag{3.16}$$

and the "homogeneous" subgroup is generated by

$$K_{x\tau}, \quad K_{y\tau}, \quad J_{xy}. \tag{3.17}$$

This group is a semidirect product, and as for the $O(4,1)$ case, its "translation" part must be trivially represented.

The translation $T_1(\alpha)$ in this case, acts on a^α, is now

$$T_1(\alpha)a_\alpha = \begin{bmatrix} a_0 - (\alpha^2/2)\delta - \alpha a_5 \\ a_1 \\ a_2 \\ a_3 - (\alpha^2/2)\delta - \alpha a_5 \\ a_5 + \alpha\delta \end{bmatrix}, \tag{3.18}$$

where δ is again (by our choice of "longitudinal" components) $a_0 - a_3$. We must therefore again have $\delta = 0$ as in Eq. (3.13) and use the fact that a_0, a_3 are gauge equivalent to $a_0 - \alpha a_5$, $a_3 - \alpha a_5$ by Eq. (3.12) [this argument is valid if we use $T_2(\beta), T_3(\gamma)$ as well].

The components a_1, a_2, a_5 remain in both the $O(4,1)$ and $O(3,2)$ cases as unaffected by local gauge transformation. In the $O(4,1)$ case, they form a unitary representation of $O(3)$ by which a unitary representation of $O(3,1)$ can be induced with orbit labelled by timelike vectors k^μ.[14] In the $O(3,2)$ case, they form a nonunitary representation of $O(2,1)$ from which a nonunitary representation of $O(3,1)$ can be induced, with orbit labelled by spacelike vectors k^μ.[15,16]

Let us return briefly to the conservation law (1.6) and the field equations (1.8). As remarked by Stueckelberg,[1] the integral over all τ of (1.6) yields

$$\partial_\mu J^\mu(x) = 0, \tag{3.19}$$

where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau \, j_\tau^\mu(x), \tag{3.20}$$

in correspondence with (1.1), where we have assumed that $\rho_\tau(x)|_{\pm\infty} = 0$ (pointwise).[5] It is clear that $J^\mu(x)$ is a candidate as the source for the Maxwell field. In fact, let us consider the integral of the field equation (1.8) over τ. Writing the $\mu = 0,1,2,3$ components as

$$\partial_\tau f^{\mu 5} + \partial_\mu f^{\mu\nu} = j^\mu, \tag{3.21}$$

if we assume that $f^{\mu 5}|_{\tau = \pm\infty} = 0$, we obtain

$$\partial_\nu f^{\mu\nu} = J^\mu, \tag{3.22}$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \, f^{\mu\nu}(x), \tag{3.23}$$

so that we may identify the integrals

$$A^\mu(x) = \int_{-\infty}^{\infty} a^\mu(x) d\tau \tag{3.24}$$

with the Maxwell vector potentials (the fields a^μ therefore have dimension L^{-2}, and the "charge" e' dimensions L).[6] The fifth component of (1.8),

$$\partial_\mu f^{5\mu}=j^5=\rho \tag{3.25}$$

has the τ-integral

$$-\partial^\mu\partial_\mu A^5=P, \tag{3.26}$$

assuming that $a^\mu|_{\tau=\pm\infty}=0$, where

$$A^5=\int_{-\infty}^{\infty} a_5\,d\tau, \quad P=\int_{-\infty}^{\infty} \rho(x)d\tau. \tag{3.27}$$

Hence, the A^5 field decouples from the two transverse polarization states.

Possible physical properties of the new pseudo-Maxwell field A^5, and the second quantization of the pre-Maxwell fields will be discussed elsewhere.[7]

IV. CONCLUSIONS AND SUMMARY

We have shown that the massless property of the free Maxwell electromagnetic field implies the structure of its polarization states and also its gauge properties. This is done through the analysis of the action on the four components of the vector potential by the little group $E(2)$ of the four-momentum. Since $E(2)$ is solvable, its finite-dimensional representations have the property that the "translations" of $E(2)$, given by Eq. (2.4), must be represented trivially. This condition, written in terms of the transformation properties of the field for a given lightlike four momentum under the Lorentz group, restricts the form of the vector potential up to the remaining gauge freedom at this four momentum. The Gupta–Bleuler condition emerges as a result of the Lorentz covariance of the theory.

Applying the same method to the five-dimensional field a^α, one finds that the stabilizer of the five "momentum" (also required for the free case to satisfy $k^\alpha k_\alpha=0$) is $E(3)$ in the case that the metric is $O(4,1)$, and $E(2,1)$ in the case that the metric is $O(3,2)$. In both cases, these simply connected Lie groups can be decomposed into a semidirect product of the translation part, which is a maximal simply connected solvable normal subgroup and a simply connected semisimple subgroup [$O(3)$ or $O(2,1)$, respectively]. If follows from a theorem of Lie[10] that in an irreducible representation of such a group, the solvable subgroup is represented by its (invariant) character. Furthermore, in a finite-dimensional representation, the character must be unity (Ref. 10, p. 227). Hence the "translations" of $E(3)$ and $E(2,1)$, of the form (3.13), must also be represented trivially. Imposing this condition on the five-vector field, we find that a Gupta–Bleuler type condition again emerges as a result of the larger $O(4,1)$ or $O(3,2)$ symmetry of the theory, and that the remaining polarization states are transverse to the k^α vector.

In both cases, one obtains a solution for the field tensor in which the "longitudinal" components are gauge equivalent to zero.

In the Maxwell limit for which integration over τ is carried out on all of the fields of the five-dimensional generalization, we find the fifth component decouples and the Lorentz vector part of the field remains with just the two Maxwell polarizations. As we pointed out for the Maxwell case, the polarization vectors provide a representation for the stability group of the Poynting vector. The method we have used therefore has implication for the structure of the Poynting vector of the five-dimensional theory, strengthened by the requirement that this theory be a proper generalization of the Maxwell theory. We shall discuss this point in detail elsewhere.

We also remark that the $O(3,2)$ case, the polarization states provide a finite-dimensional representation of $O(2,1)$ and therefore appear to present a problem in the construction of the second quantized theory in a unitary Fock space (with positive norm). The construction of these states, along lines followed in the treatment of the two body relativistic bound state problem[16] will also be discussed elsewhere.

ACKNOWLEDGMENTS

We wish to thank L. Michel and M. Marinov for discussions of this problem, and S. Weinberg for a communication on this subject. One of us (L.H.) would like to thank Professor B. Gruber for the opportunity to present an early form of this work at the symposium on Symmetries in Science VI, August 1992, in Bregenz, and to his colleagues there for discussions. He also wishes to thank S. L. Adler for his hospitality at the Institute for Advanced Study where this work was completed, and to the Monell Foundation for support during that period.

[1] E. C. G. Stueckelberg, Helv. Phys. Acta. **14**, 372, 588 (1941); **15**, 23 (1942). See also, R. P. Feynman, Phys. Rev. **80**, 440 (1980) and J. Schwinger, *ibid.* **82**, 664 (1951).
[2] L. P. Horwitz and C. Piron, Helv. Phys. Acta. **46**, 316 (1973). See also, J. R. Fanchi, Phys. Rev. D **80**, 3108 (1979); D. Dewdney, P. R. Holland, A. Kyprianidis, and J. P. Vigier, Phys. Lett. A **113**, 359 (1986); **114**, 440 (1986); and A. Kyprianidis, Phys. Rep. **155**, 1 (1987).
[3] R. P. Feynman, Phys. Rev. **74**, 939 (1948).
[4] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 612.
[5] R. Arshansky, L. P. Horwitz, and Y. Lavie, Found Phys. **13**, 1167 (1983).
[6] D. Saad, L. P. Horwitz, and R. I. Arshansky, Found Phys. **19**, 1125 (1989).
[7] N. Shnerb and L. P. Horwitz, Phys. Rev. A **48**, 4098 (1993).
[8] S. Weinberg, Phys. Rev. B **138**, 988 (1965); **134**, 882 (1964). See also *Lectures on Particles and Field Theory [Brandeis Summer Institute in Theoretical Physics (1964)]*, edited by S. Deser and K. W. Ford, Vol. 2, p. 405 (Prentice Hall, Englewood Cliffs, 1965).
[9] R. Shaw, Nuovo Cimento. **37**, 1086 (1965).
[10] A. O. Barut and R. Raczka, *Theory of Group Representations and Application* (PWN-Polish Scientific, Warsaw, 1977), p. 200.
[11] E. P. Wigner, in *Theoretical Physics* (International Atomic Energy Agency, Vienna 1963), p. 59.
[12] A. O. Barut and R. Raczka, Inst. Henri Poincaré **XVII**, 111 (1972).
[13] See, for example, F. Mandl, *Introduction to Quantum Field Theory* (Interscience, New York, 1959).
[14] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[15] V. Bergman, Ann. Math. **48**, 568 (1944).
[16] R. I. Arshansky and L. P. Horwitz, J. Math. Phys. **30**, 66 and 380 (1989).